

Article

Soft Limit and Soft Continuity

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Abstract: This study presents the soft limit and upper (lower) soft limit proposed by Molodtsov, with several theoretical contributions. It investigates some of their basic properties, such as some fundamental soft limit rules, the relation between soft limit and boundedness, and the sandwich/squeeze theorem. Moreover, the paper proposes left and right soft limits and studies some of their main properties. Furthermore, it defines the soft limit at infinity and explores some of its basic properties. Additionally, the present study exemplifies these concepts and their properties to better understand them. The paper then compares the aforesaid concepts with their classical forms. Afterward, this paper presents soft continuity and upper (lower) soft continuity, proposed by Molodtsov, theoretically contributes to these concepts, and investigates some of their key properties, such as some fundamental soft continuity rules, the relation between soft continuity and boundedness, Bolzano's theorem, and the intermediate value theorem. Moreover, it defines left and right soft continuity and studies some of their basic properties. The present study exemplifies soft continuity types and their properties. In addition, it compares them with their classical forms. Finally, this study discusses whether the aspects should be further analyzed.

Keywords: soft sets; soft analysis; soft limit; soft continuity

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1. Introduction

Soft set theory, first introduced by Molodtsov [1], offers a versatile mathematical framework for handling uncertainty. This theory has led to the emergence of soft analysis, where fundamental analytical notions, such as limit and continuity, are reconsidered through the lens of soft sets. Despite their foundational significance, Molodtsov's initial investigation of soft limit and soft continuity is limited in scope, and subsequent works—including [2–6], published exclusively in Russian—have yet to secure these notions a place within the broader mathematical mainstream.

Soft limit and soft continuity form the groundwork for developing other analytical concepts in soft set theory, including soft derivative, soft integral, and soft differential equations [1,7–10]. Therefore, a deeper investigation into their properties is not only mathematically significant but also essential for improving the theory of soft analysis, which has not been satisfactorily attended to in the related literature. The present paper addresses this research need by extending and systematizing the theory of soft limit and soft continuity by introducing left and right soft limits, soft left and right variances, the soft limit at infinity, and left and right soft continuity and by redefining the concepts of soft and upper (lower) soft limits and soft and upper (lower) soft continuity to establish a coherent framework.

In addition to these theoretical developments, this study contributes to the literature by offering characterizations of soft limit and soft continuity types and by investigating their properties, as well as establishing their relationships with their counterparts in classical analysis. Classical limit and classical continuity underpin a vast array of applications—ranging from instantaneous velocity and acceleration in mechanics to smooth input-output mappings in control systems and signal processing—from marginal cost and revenue analyses in economics to the formulation and solution of differential equations and the definition of continuous probability distributions; yet, they rest on idealized assumptions of infinite divisibility and perfect measurement precision. As Molodtsov et al. [5] have noted, the impossibility of infinite partitioning of matter, combined with fundamental quantum indeterminacy and the limitations of physical measurement, undermines these assumptions and originates a need for analytical tools capable of handling inherent uncertainty and approximation. By redefining the soft limit and soft continuity, including upper and lower soft limits and upper and lower soft continuity, as well as introducing left and right soft limits, the soft limit at infinity, and left and right continuity, this paper not only strengthens the theoretical foundations of soft analysis but also paves the way for such applications as quantum-informed physical modeling and imprecise economic systems, where classical methods prove inadequate.

The remainder of this paper is organized as follows: Section 2 presents some basic definitions and properties to be needed in the following sections. Section 3 provides the soft and upper (lower) soft limits in [1,3–5], along with several theoretical contributions. Moreover, it defines left and right soft limits and studies some of their basic properties. Additionally, it exemplifies these new concepts since they are not widely known. Section 4 introduces the soft limit at infinity and explores some of their basic properties. Section 5 compares soft limit types with their classical counterparts. Section 6 presents soft and upper (lower) continuity in [3–5] with several theoretical contributions, proposes left and right soft continuity, and illustrates these concepts. Section 7 investigates the relationships between concepts in Section 6 and their classical forms. The final section inquires whether further research should be conducted or not.

2. Preliminaries

This section presents some basic definitions and properties to be used in the following sections. Throughout this paper, the notations $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^-$, $\mathbb{R}^+$, and $\mathbb{R}^{\geq 0}$ represent the set of all nonnegative integers, real numbers, negative real numbers, positive real numbers, and nonnegative real numbers, respectively. For any set U , let $P(U)$ be the set of all classical subsets of U , i.e., the power set of U .

Definition 1 ([1]). Let U be a universal set, E be a parameter set, and $f : E \rightarrow P(U)$ be a function. Then, (f, E) (briefly f) is called a soft set parameterized via E over U (briefly a soft set over U).

Example 1. The function $f : \mathbb{N} \rightarrow P(\mathbb{R})$ defined by $f(x) = [x, x + 1]$ is a soft set over $\mathbb{R}$.

Definition 2 ([1,3]). Let M be a set called a model set, U be a universal set, E be a parameter set, and $f : M \times E \rightarrow P(U)$ be a function. Then, f is called a soft mapping parameterized via $M \times E$ over U (briefly a soft mapping over U).

3. Soft Limit

This section presents and theoretically contributes the soft limit of real-valued functions with one real variable provided in [1,3–5]. Throughout this section, let $\tau, \lambda, \kappa : \mathbb{R} \rightarrow P(\mathbb{R})$, $\varepsilon, \alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$, and $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be seven functions such that $\tau(a)$, $\lambda(a)$, and $\kappa(a)$ are sets of points close to a but not equal to a . Moreover,

across this study, let $\Phi(A, B) := \{f \mid f : A \rightarrow B \text{ is a function}\}$ and let $\tau_{D(f)}(a)$ denote $\tau(a) \cap \text{Dom}(f)$. Here, $\text{Dom}(f)$ is the domain set of f . In addition, $\tilde{\tau}(a) := \tau(a) \cup \{a\}$ and thus $\tilde{\tau}_{D(f)}(a) := \tilde{\tau}(a) \cap \text{Dom}(f)$, for all $a \in \mathbb{R}$. These descriptions also hold for λ and κ . Throughout this section, let $A, B \subseteq \mathbb{R}$, $a, t, L \in \mathbb{R}$, and $f, g, h : A \rightarrow \mathbb{R}$ be three functions, unless stated otherwise.

Definition 3 ([3]). The set of all points belonging to $\tau(a)$ and are greater than a is defined by $\tau^+(a) := \tau(a) \cap (a, \infty)$, and the set of all points belonging to $\tau(a)$ and are less than a is defined by $\tau^-(a) := \tau(a) \cap (-\infty, a)$. Moreover, if $\tau^-(a) = \emptyset$, for all $a \in \mathbb{R}$, then τ is called a right mapping, and if $\tau^+(a) = \emptyset$, for all $a \in \mathbb{R}$, then τ is called a left mapping. Furthermore, $\tau_\delta(a)$ is defined by $\tau_\delta(a) := [a - \delta(a), a] \cup (a, a + \delta(a)]$. Therefore, $\tau_\delta^+(a) = (a, a + \delta(a)]$ and $\tau_\delta^-(a) = [a - \delta(a), a]$. In addition, $\tilde{\tau}^+(a) = \tau^+(a) \cup \{a\}$ and $\tilde{\tau}^-(a) = \tau^-(a) \cup \{a\}$, for all $a \in \mathbb{R}$. Additionally, $\tilde{\tau}_\delta(a) = \tau_\delta(a) \cup \{a\} = [a - \delta(a), a + \delta(a)]$ and thus $\tilde{\tau}_\delta^+(a) = [a, a + \delta(a)]$ and $\tilde{\tau}_\delta^-(a) = [a - \delta(a), a]$, for all $a \in \mathbb{R}$.

Note 1. It can be observed from Definition 3 that $\tau(a) \subseteq \tilde{\tau}(a)$, $\tau^+(a) \subseteq \tilde{\tau}^+(a)$, $\tau^-(a) \subseteq \tilde{\tau}^-(a)$, $\tau(a) = \tau^+(a) \cup \tau^-(a)$, $\tau_\delta(a) = \tau_\delta^+(a) \cup \tau_\delta^-(a)$, $\tilde{\tau}(a) = \tilde{\tau}^+(a) \cup \tilde{\tau}^-(a)$, and $\tilde{\tau}_\delta(a) = \tilde{\tau}_\delta^+(a) \cup \tilde{\tau}_\delta^-(a)$, for all $a \in \mathbb{R}$. Moreover, $\tau_{D(f)}^+(a) := \tau^+(a) \cap \text{Dom}(f)$, $\tau_{D(f)}^-(a) := \tau^-(a) \cap \text{Dom}(f)$, $\tilde{\tau}_{D(f)}^+(a) = \tilde{\tau}^+(a) \cap \text{Dom}(f)$, and $\tilde{\tau}_{D(f)}^-(a) = \tilde{\tau}^-(a) \cap \text{Dom}(f)$, for all $a \in \mathbb{R}$. Additionally, if $\delta(a) = k$, for all $a \in \mathbb{R}$, i.e., δ is a constant function, then the notation τ_k and $\tilde{\tau}_k$ can be used instead of the notations τ_δ and $\tilde{\tau}_\delta$, respectively.

The descriptions in Definition 3 and Note 1 also hold for λ and κ .

Definition 4 ([1,4]). L is called a (τ, ε) -soft limit of f at a if

$$x \in \tau(a) \Rightarrow (x \in A \wedge |f(x) - L| \leq \varepsilon(a))$$

The set of all (τ, ε) -soft limits of f at a is denoted by $\text{slim}(f, a, \tau, \varepsilon)$. If $\text{slim}(f, a, \tau, \varepsilon) = \emptyset$, then the (τ, ε) -soft limit of f at a does not exist.

Definition 5 ([1,4]). L is called an upper (τ, ε) -soft limit of f at a if $x \in \tau(a) \Rightarrow (x \in A \wedge f(x) \leq L + \varepsilon(a))$. The set of all upper (τ, ε) -soft limits of f at a is denoted by $\overline{\text{slim}}(f, a, \tau, \varepsilon)$. If $\overline{\text{slim}}(f, a, \tau, \varepsilon) = \emptyset$, then the upper (τ, ε) -soft limit of f at a does not exist.

Definition 6 ([1,4]). L is called a lower (τ, ε) -soft limit of f at a if $x \in \tau(a) \Rightarrow (x \in A \wedge L - \varepsilon(a) \leq f(x))$. The set of all lower (τ, ε) -soft limits of f at a is denoted by $\underline{\text{slim}}(f, a, \tau, \varepsilon)$. If $\underline{\text{slim}}(f, a, \tau, \varepsilon) = \emptyset$, then the lower (τ, ε) -soft limit of f at a does not exist.

Note 2. Molodtsov defines the concept of (τ, ε) -soft limit in [2] as follows: "Let $\tau_{D(f)}(a) \neq \emptyset$. Then, L is called a (τ, ε) -soft limit of f at a if $\sup_{x \in \tau_{D(f)}(a)} \{|f(x) - L|\} \leq \varepsilon(a)$. The set of all (τ, ε) -soft limits of f at a is denoted by $\text{slim}(f, a, \tau, \varepsilon)$. If $\text{slim}(f, a, \tau, \varepsilon) = \emptyset$, then the (τ, ε) -soft limit of f at a does not exist." Here, the statement $\sup_{x \in \tau_{D(f)}(a)} \{|f(x) - L|\} \leq \varepsilon(a)$ is equivalent to the statement $x \in \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$. Therefore, this study uses Definitions 4–6 as in Definitions 7–9, respectively. Furthermore, it readjusts the related properties provided in [1,3–5] according to Definitions 7–9.

Definition 7. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, L is called a (τ, ε) -soft limit of f at a if

$$x \in \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$$

The set of all (τ, ε) -soft limits of f at a is denoted by $\text{slim}(f, a, \tau, \varepsilon)$. If $\text{slim}(f, a, \tau, \varepsilon) = \emptyset$, then the (τ, ε) -soft limit of f at a does not exist.

Definition 8. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, L is called an upper (τ, ε) -soft limit of f at a if $x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq L + \varepsilon(a)$. The set of all upper (τ, ε) -soft limits of f at a is denoted by $\overline{\text{slim}}(f, a, \tau, \varepsilon)$. If $\overline{\text{slim}}(f, a, \tau, \varepsilon) = \emptyset$, then the upper (τ, ε) -soft limit of f at a does not exist.

Definition 9. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, L is called a lower (τ, ε) -soft limit of f at a if $x \in \tau_{D(f)}(a) \Rightarrow L - \varepsilon(a) \leq f(x)$. The set of all lower (τ, ε) -soft limits of f at a is denoted by $\underline{\text{slim}}(f, a, \tau, \varepsilon)$. If $\underline{\text{slim}}(f, a, \tau, \varepsilon) = \emptyset$, then the lower (τ, ε) -soft limit of f at a does not exist.

In addition, the left (τ, ε) -soft limit and right (τ, ε) -soft limit are defined as follows:

Definition 10. Let $\tau_{D(f)}^-(a) \neq \emptyset$. Then, L is called a left (τ, ε) -soft limit of f at a if $x \in \tau_{D(f)}^-(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$. The set of all left (τ, ε) -soft limits of f at a is denoted by $\text{slim}(f, a, \tau^-, \varepsilon)$. If $\text{slim}(f, a, \tau^-, \varepsilon) = \emptyset$, then the left (τ, ε) -soft limit of f at a does not exist.

Definition 11. Let $\tau_{D(f)}^+(a) \neq \emptyset$. Then, L is called a right (τ, ε) -soft limit of f at a if $x \in \tau_{D(f)}^+(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$. The set of all right (τ, ε) -soft limits of f at a is denoted by $\text{slim}(f, a, \tau^+, \varepsilon)$. If $\text{slim}(f, a, \tau^+, \varepsilon) = \emptyset$, then the right (τ, ε) -soft limit of f at a does not exist.

Example 2. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x + 3$, $a = 2$, $\delta(2) = \frac{1}{2}$, $\varepsilon(2) = 2$, and $\tau_\delta(2) = [2 - \delta(2), 2) \cup (2, 2 + \delta(2)] = [\frac{3}{2}, 2) \cup (2, \frac{5}{2}]$,

$$\begin{aligned} \forall x \in \tau_\delta(2) \cap \mathbb{R}, |f(x) - L| \leq \varepsilon(2) &\Leftrightarrow \forall x \in \tau_\delta(2) \cap \mathbb{R}, |2x + 3 - L| \leq 2 \\ &\Leftrightarrow \forall x \in \tau_\delta(2) \cap \mathbb{R}, 1 + 2x \leq L \leq 5 + 2x \\ &\Leftrightarrow L \in [6, 8] \end{aligned}$$

Therefore, $\text{slim}(f, 2, \tau_\delta, \varepsilon) = [6, 8]$. For example, if $L = 7$, then

$$\begin{aligned} x \in \tau_\delta(2) \cap \mathbb{R} &\Rightarrow x \in ([\frac{3}{2}, 2) \cup (2, \frac{5}{2}]) \cap \mathbb{R} \\ &\Rightarrow 0 < |x - 2| \leq \frac{1}{2} \\ &\Rightarrow |2x - 4| \leq 1 \\ &\Rightarrow |2x + 3 - 7| \leq 2 \end{aligned}$$

That is, $L = 7$ is a $(\tau_\delta, \varepsilon)$ -soft limit of f at $a = 2$. Similarly, $\overline{\text{slim}}(f, 2, \tau_\delta, \varepsilon) = [6, \infty)$, $\underline{\text{slim}}(f, 2, \tau_\delta, \varepsilon) = (-\infty, 8]$, $\text{slim}(f, 2, \tau_\delta^-, \varepsilon) = [5, 8]$, and $\text{slim}(f, 2, \tau_\delta^+, \varepsilon) = [6, 9]$.

Remark 1. It follows from Example 2 that the classical limit of a function can be a soft limit of the function (for further details, see Theorem 61).

Note 3. Each of the concepts of (τ, ε) -soft limit, upper (τ, ε) -soft limit, lower (τ, ε) -soft limit, left (τ, ε) -soft limit, and right (τ, ε) -soft limit is a soft mapping parameterized via $\Phi(A, \mathbb{R}) \times \mathbb{R} \times \Phi(\mathbb{R}, P(\mathbb{R})) \times \Phi(\mathbb{R}, \mathbb{R}^{\geq 0})$ over $\mathbb{R}$ such that $\emptyset \neq A \subseteq \mathbb{R}$.

Definition 12 ([3]). If $\sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \in \mathbb{R}$, then this real number is called soft τ -variance of f at a and denoted by $\Delta(f, a, \tau)$.

Definition 13. If $\sup_{x,y \in \tau_{D(f)}^-(a)} \{f(x) - f(y)\} \in \mathbb{R}$, then this real number is called soft τ -left variance of f at a and denoted by $\Delta(f, a, \tau^-)$.

Definition 14. If $\sup_{x,y \in \tau_{D(f)}^+(a)} \{f(x) - f(y)\} \in \mathbb{R}$, then this real number is called soft τ -right variance of f at a and denoted by $\Delta(f, a, \tau^+)$.

Example 3. For $\tau_3(2) = [-1, 2) \cup (2, 5]$, the soft τ_3 -variance of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$ at $a = 2$ is as follows:

$$\Delta(f, 2, \tau_3) = \sup_{x,y \in \tau_3(2) \cap \mathbb{R}} \{f(x) - f(y)\} = \sup_{x,y \in \tau_3(2) \cap \mathbb{R}} \{x - y\} = 6$$

Moreover, $\Delta(f, 2, \tau_3^-) = 3$ and $\Delta(f, a, \tau_3^+) = 3$. Here, $\tau_3^-(2) = [-1, 2)$ and $\tau_3^+(2) = (2, 5]$.

Theorem 1 ([3]). Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ if and only if (iff) $\Delta(f, a, \tau) \leq 2\varepsilon(a)$.

Corollary 1 (Cauchy's Criterion). Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $|f(x) - f(y)| \leq 2\varepsilon(a)$, for all $x, y \in \tau_{D(f)}(a)$.

Theorem 2. Let $\tau_{D(f)}^-(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$ iff $\Delta(f, a, \tau^-) \leq 2\varepsilon(a)$.

Theorem 3. Let $\tau_{D(f)}^+(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$ iff $\Delta(f, a, \tau^+) \leq 2\varepsilon(a)$.

Theorem 4 ([4]). Let $\tau_{D(f)}(a) \neq \emptyset$. If $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then f is bounded above on $\tau_{D(f)}(a)$.

Theorem 5. Let $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded above on $\tau_{D(f)}(a)$, then $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$.

Proof. Let $\tau_{D(f)}(a) \neq \emptyset$ and f be bounded above on $\tau_{D(f)}(a)$. Then, $\sup_{\tau_{D(f)}(a)} f \in \mathbb{R}$. Thus, $f(x) \leq \sup_{\tau_{D(f)}(a)} f \leq \sup_{\tau_{D(f)}(a)} f + \varepsilon(a)$, for all $x \in \tau_{D(f)}(a)$ and for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$. Therefore, $\sup_{\tau_{D(f)}(a)} f \in \overline{\text{slim}}(f, a, \tau, \varepsilon)$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$. Consequently, $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$. $\square$

Corollary 2. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff f is bounded above on $\tau_{D(f)}(a)$.

Theorem 6 ([4]). Let $\tau_{D(f)}(a) \neq \emptyset$. If $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then f is bounded below on $\tau_{D(f)}(a)$.

Theorem 7. Let $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded below on $\tau_{D(f)}(a)$, then $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$.

Corollary 3. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff f is bounded below on $\tau_{D(f)}(a)$.

Theorem 8. Let $\tau_{D(f)}^-(a) \neq \emptyset$. If $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$, then f is bounded on $\tau_{D(f)}^-(a)$.

Theorem 9. Let $\tau_{D(f)}^-(a) \neq \emptyset$. If f is bounded on $\tau_{D(f)}^-(a)$, then $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$.

Corollary 4. Let $\tau_{D(f)}^-(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$ iff f is bounded on $\tau_{D(f)}^-(a)$.

Theorem 10. Let $\tau_{D(f)}^+(a) \neq \emptyset$. If $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$, then f is bounded on $\tau_{D(f)}^+(a)$.

Theorem 11. Let $\tau_{D(f)}^+(a) \neq \emptyset$. If f is bounded on $\tau_{D(f)}^+(a)$, then $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$.

Corollary 5. Let $\tau_{D(f)}^+(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$ iff f is bounded on $\tau_{D(f)}^+(a)$.

Theorem 12 ([3]). If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, then

$$\text{slim}(f, a, \tau, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right]$$

Theorem 13. If $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$, then

$$\text{slim}(f, a, \tau^-, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right]$$

Proof. Let $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$. Therefore, for all $L \in \text{slim}(f, a, \tau^-, \varepsilon)$,

$$\begin{aligned} x \in \tau_{D(f)}^-(a) &\Rightarrow |f(x) - L| \leq \varepsilon(a) \\ &\Rightarrow L - \varepsilon(a) \leq f(x) \leq L + \varepsilon(a) \end{aligned}$$

Thereby,

$$\begin{aligned} L - \varepsilon(a) \leq f(x) &\Rightarrow \inf_{x \in \tau_{D(f)}^-(a)} \{L - \varepsilon(a)\} \leq \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} \\ &\Rightarrow L \leq \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \end{aligned}$$

and

$$\begin{aligned} f(x) \leq L + \varepsilon(a) &\Rightarrow \sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} \leq \sup_{x \in \tau_{D(f)}^+(a)} \{L + \varepsilon(a)\} \\ &\Rightarrow \sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} - \varepsilon(a) \leq L \end{aligned}$$

Thus,

$$\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a) \leq L \leq \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a)$$

Hence,

$$\text{slim}(f, a, \tau^-, \varepsilon) \subseteq \left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right]$$

Moreover, for all $L \in \left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right]$,

$$\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a) \leq L \leq \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a)$$

From the definitions of supremum and infimum,

$$\begin{aligned} x \in \tau_{D(f)}^-(a) &\Rightarrow f(x) - \varepsilon(a) \leq L \leq f(x) + \varepsilon(a) \\ &\Rightarrow |f(x) - L| \leq \varepsilon(a) \end{aligned}$$

Thus,

$$\left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right] \subseteq \text{slim}(f, a, \tau^-, \varepsilon)$$

Consequently,

$$\text{slim}(f, a, \tau^-, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right]$$

□

Theorem 14. If $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$, then

$$\text{slim}(f, a, \tau^+, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^+(a)} \{f(x)\} + \varepsilon(a) \right]$$

The proof of Theorem 14 is similar to the proof of Theorem 13.

Theorem 15 ([4]). If $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then

$$\overline{\text{slim}}(f, a, \tau, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \infty \right)$$

Theorem 16 ([4]). If $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then

$$\underline{\text{slim}}(f, a, \tau, \varepsilon) = \left(-\infty, \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right]$$

Theorem 17. $\text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$ iff $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Moreover,

$$\text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon) = \text{slim}(f, a, \tau, \varepsilon)$$

Proof. ($\Rightarrow$): Let $\text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$. Then, there exists an $L \in \text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon)$ such that

$$x \in \tau_{D(f)}^-(a) \Rightarrow |f(x) - L| \leq \varepsilon(a) \quad \text{and} \quad x \in \tau_{D(f)}^+(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$$

Thus,

$$x \in \tau_{D(f)}^-(a) \cup \tau_{D(f)}^+(a) = \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$$

Hence, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$.

($\Leftarrow$): Let $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Then, there exists an $L \in \text{slim}(f, a, \tau, \varepsilon)$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$$

Hence,

$$\begin{aligned} x \in \tau_{D(f)}^-(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow |f(x) - L| \leq \varepsilon(a) \end{aligned}$$

and

$$\begin{aligned} x \in \tau_{D(f)}^+(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow |f(x) - L| \leq \varepsilon(a) \end{aligned}$$

Therefore, $L \in \text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon)$, i.e., $\text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$.

Moreover,

$$\begin{aligned} \text{slim}(f, a, \tau^-, \varepsilon) \cap \text{slim}(f, a, \tau^+, \varepsilon) &= \left[\sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a) \right] \cap \\ &\quad \left[\sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}^+(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= \left[\max \left\{ \sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} - \varepsilon(a) \right\}, \right. \\ &\quad \left. \min \left\{ \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a), \inf_{x \in \tau_{D(f)}^+(a)} \{f(x)\} + \varepsilon(a) \right\} \right] \\ &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= \text{slim}(f, a, \tau, \varepsilon) \end{aligned}$$

Here,

$$\begin{aligned} \max \left\{ \sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\} - \varepsilon(a), \sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} - \varepsilon(a) \right\} &= \max \left\{ \sup_{x \in \tau_{D(f)}^-(a)} \{f(x)\}, \sup_{x \in \tau_{D(f)}^+(a)} \{f(x)\} \right\} - \varepsilon(a) \\ &= \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a) \end{aligned}$$

and

$$\begin{aligned} \min \left\{ \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\} + \varepsilon(a), \inf_{x \in \tau_{D(f)}^+(a)} \{f(x)\} + \varepsilon(a) \right\} &= \min \left\{ \inf_{x \in \tau_{D(f)}^-(a)} \{f(x)\}, \inf_{x \in \tau_{D(f)}^+(a)} \{f(x)\} \right\} + \varepsilon(a) \\ &= \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \end{aligned}$$

□

Theorem 18. $\overline{\text{slim}}(f, a, \tau, \varepsilon) \cap \underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Moreover,

$$\overline{\text{slim}}(f, a, \tau, \varepsilon) \cap \underline{\text{slim}}(f, a, \tau, \varepsilon) = \text{slim}(f, a, \tau, \varepsilon)$$

Corollary 6. Let $\tau_{D(f)}(a) \neq \emptyset$. If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, then f is bounded on $\tau_{D(f)}(a)$.

Corollary 7. Let $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded on $\tau_{D(f)}(a)$, then $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$.

Corollary 8. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff f is bounded on $\tau_{D(f)}(a)$.

Example 4. For the function $f : A \rightarrow \mathbb{R}$ defined by $f(x) = c$ such that $\tau_{D(f)}(a) \neq \emptyset$. Since $\Delta(f, a, \tau) = \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} = \sup_{x, y \in \tau_{D(f)}(a)} \{c - c\} = 0 \leq 2\varepsilon(a)$, then there exists the (τ, ε) -soft limit of f at a .
Moreover,

$$\text{slim}(f, a, \tau, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] = [c - \varepsilon(a), c + \varepsilon(a)]$$

To exemplify, for the function $f : \mathbb{N} \rightarrow \mathbb{R}$ defined by $f(x) = 5$, $\tau_2(3) = [1, 3) \cup (3, 5]$, and $\varepsilon(3) = \frac{1}{2}$,

$$\text{slim}(f, 3, \tau_2, \varepsilon) = \left[\sup_{x \in \tau_2(3) \cap \mathbb{N}} \{f(x)\} - \frac{1}{2}, \inf_{x \in \tau_2(3) \cap \mathbb{N}} \{f(x)\} + \frac{1}{2} \right] = \left[5 - \frac{1}{2}, 5 + \frac{1}{2} \right] = \left[\frac{9}{2}, \frac{11}{2} \right]$$

where $\tau_2(3) \cap \mathbb{N} = \{1, 2, 4, 5\}$.

Example 5. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$ and $\varepsilon(2) = 4$,

$$\begin{aligned} \forall x \in \tau_3(2) \cap \mathbb{R}, |x + 3 - L| \leq 4 &\Leftrightarrow \forall x \in \tau_3(2) \cap \mathbb{R}, -4 \leq x + 3 - L \leq 4 \\ &\Leftrightarrow \forall x \in \tau_3(2) \cap \mathbb{R}, -1 + x \leq L \leq 7 + x \\ &\Leftrightarrow L \in [4, 6] \end{aligned}$$

Thus, $\text{slim}(f, 2, \tau_3, \varepsilon) = [4, 6]$. Moreover, from Theorem 12,

$$\text{slim}(f, 2, \tau_3, \varepsilon) = \left[\sup_{x \in \tau_3(2) \cap \mathbb{R}} \{x + 3\} - 4, \inf_{x \in \tau_3(2) \cap \mathbb{R}} \{x + 3\} + 4 \right] = [4, 6]$$

Furthermore, $\overline{\text{slim}}(f, 2, \tau_3, \varepsilon) = [4, \infty)$, $\underline{\text{slim}}(f, 2, \tau_3, \varepsilon) = (-\infty, 6]$, $\text{slim}(f, 2, \tau_3^-, \varepsilon) = [1, 6]$, and $\text{slim}(f, 2, \tau_3^+, \varepsilon) = [4, 9]$. Hence,

$$\overline{\text{slim}}(f, 2, \tau_3, \varepsilon) \cap \underline{\text{slim}}(f, 2, \tau_3, \varepsilon) = \text{slim}(f, 2, \tau_3, \varepsilon) = \text{slim}(f, 2, \tau_3^-, \varepsilon) \cap \text{slim}(f, 2, \tau_3^+, \varepsilon)$$

Example 6. For the function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, there is no classical limit of f at $a = 0$. Moreover, for $\varepsilon(0) \geq 0$ and $\delta(0) > 0$,

$$\begin{aligned} \text{slim}(f, 0, \tau_\delta, \varepsilon) &= \left[\sup_{x \in \tau_\delta(0) \cap (\mathbb{R} \setminus \{0\})} \left\{ \frac{1}{x} \right\} - \varepsilon(0), \inf_{x \in \tau_\delta(0) \cap (\mathbb{R} \setminus \{0\})} \left\{ \frac{1}{x} \right\} + \varepsilon(0) \right] \\ &= [\infty, -\infty] = \emptyset \end{aligned}$$

Theorem 19. Let $b \in A$ and $a \neq b$. Then, there exists a function $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ such that $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$.

Proof. Let $b \in A$ and $a \neq b$. For a function $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ such that $\tau_{D(f)}(a) = \{b\}$,

$$\begin{aligned} \text{slim}(f, a, \tau, \varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= [f(b) - \varepsilon(a), f(b) + \varepsilon(a)] \neq \emptyset \end{aligned}$$

□

Note 4. Example 6 manifests the existence of a function with neither classical nor $(\tau_\delta, \varepsilon)$ -soft limit, for all $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ and $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$. On the other hand, for the function f in Example 6 and a function $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ such that $\tau_{D(f)}(0)$ is a single point set, e.g., $\tau_{D(f)}(0) = \left\{ \frac{\delta(0)}{2} \right\}$,

$$\begin{aligned} \text{slim}(f, 0, \tau, \varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(0)} \{f(x)\} - \varepsilon(0), \inf_{x \in \tau_{D(f)}(0)} \{f(x)\} + \varepsilon(0) \right] \\ &= \left[f\left(\frac{\delta(0)}{2}\right) - \varepsilon(0), f\left(\frac{\delta(0)}{2}\right) + \varepsilon(0) \right] \neq \emptyset \end{aligned}$$

Theorem 20. Let $\beta(a) \leq \alpha(a)$. If $\text{slim}(f, a, \tau, \beta) \neq \emptyset$, then $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \beta) \subseteq \text{slim}(f, a, \tau, \alpha)$.

Proof. Let $\beta(a) \leq \alpha(a)$ and $\text{slim}(f, a, \tau, \beta) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned} x \in \tau_{D(f)}(a) &\Rightarrow |f(x) - L| \leq \beta(a) \\ &\Rightarrow |f(x) - L| \leq \beta(a) \leq \alpha(a) \end{aligned}$$

Therefore, $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$. Moreover, since

$$\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \leq \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \beta(a)$$

and

$$\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \beta(a) \leq \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a)$$

then

$$\begin{aligned} \text{slim}(f, a, \tau, \beta) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \beta(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \beta(a) \right] \\ &\subseteq \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right] \\ &= \text{slim}(f, a, \tau, \alpha) \end{aligned}$$

□

Theorem 21. Let $\beta(a) \leq \alpha(a)$. If $\text{slim}(f, a, \tau^-, \beta) \neq \emptyset$, then $\text{slim}(f, a, \tau^-, \alpha) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^-, \beta) \subseteq \text{slim}(f, a, \tau^-, \alpha)$.

Theorem 22. Let $\beta(a) \leq \alpha(a)$. If $\text{slim}(f, a, \tau^+, \beta) \neq \emptyset$, then $\text{slim}(f, a, \tau^+, \alpha) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^+, \beta) \subseteq \text{slim}(f, a, \tau^+, \alpha)$.

Theorem 23. Let $\beta(a) \leq \alpha(a)$. If $\overline{\text{slim}}(f, a, \tau, \beta) \neq \emptyset$, then $\overline{\text{slim}}(f, a, \tau, \alpha) \neq \emptyset$. Moreover, $\overline{\text{slim}}(f, a, \tau, \beta) \subseteq \overline{\text{slim}}(f, a, \tau, \alpha)$.

Theorem 24. Let $\beta(a) \leq \alpha(a)$. If $\underline{\text{slim}}(f, a, \tau, \beta) \neq \emptyset$, then $\underline{\text{slim}}(f, a, \tau, \alpha) \neq \emptyset$. Moreover, $\underline{\text{slim}}(f, a, \tau, \beta) \subseteq \underline{\text{slim}}(f, a, \tau, \alpha)$.

The proofs of Theorems 21–24 are similar to the proof of Theorem 20.

Theorem 25. Let $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$. If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, then $\text{slim}(f, a, \lambda, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \varepsilon) \subseteq \text{slim}(f, a, \lambda, \varepsilon)$.

Proof. Let $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$ and $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that $x \in \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$. Thus,

$$\begin{aligned} x \in \lambda_{D(f)}(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow |f(x) - L| \leq \varepsilon(a) \end{aligned}$$

Therefore, $\text{slim}(f, a, \lambda, \varepsilon) \neq \emptyset$. Moreover, since $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$, then

$$\sup_{x \in \lambda_{D(f)}(a)} \{f(x)\} - \varepsilon(a) \leq \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a)$$

and

$$\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \leq \inf_{x \in \lambda_{D(f)}(a)} \{f(x)\} + \varepsilon(a)$$

Hence,

$$\begin{aligned} \text{slim}(f, a, \tau, \varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &\subseteq \left[\sup_{x \in \lambda_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \lambda_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= \text{slim}(f, a, \lambda, \varepsilon) \end{aligned}$$

□

Theorem 26. Let $\emptyset \neq \lambda_{D(f)}^-(a) \subseteq \tau_{D(f)}^-(a)$. If $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$, then $\text{slim}(f, a, \lambda^-, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^-, \varepsilon) \subseteq \text{slim}(f, a, \lambda^-, \varepsilon)$.

Theorem 27. Let $\emptyset \neq \lambda_{D(f)}^+(a) \subseteq \tau_{D(f)}^+(a)$. If $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$, then $\text{slim}(f, a, \lambda^+, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^+, \varepsilon) \subseteq \text{slim}(f, a, \lambda^+, \varepsilon)$.

Theorem 28. Let $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$. If $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then $\overline{\text{slim}}(f, a, \lambda, \varepsilon) \neq \emptyset$. Moreover, $\overline{\text{slim}}(f, a, \tau, \varepsilon) \subseteq \overline{\text{slim}}(f, a, \lambda, \varepsilon)$.

Theorem 29. Let $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$. If $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$, then $\underline{\text{slim}}(f, a, \lambda, \varepsilon) \neq \emptyset$. Moreover, $\underline{\text{slim}}(f, a, \tau, \varepsilon) \subseteq \underline{\text{slim}}(f, a, \lambda, \varepsilon)$.

The proofs of Theorems 26–29 are similar to the proof of Theorem 25.

Theorem 30 (Constant Multiple Rule). Let $t \neq 0$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, a, \tau, |t|\varepsilon) \neq \emptyset$. Moreover,

$$\text{slim}(tf, a, \tau, |t|\varepsilon) = t \text{slim}(f, a, \tau, \varepsilon)$$

Proof. Let $t \neq 0$.

($\Rightarrow$): Let $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Then, $\Delta(f, a, \tau) = \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \leq 2\varepsilon(a)$. Therefore,

$$\begin{aligned} \Delta(tf, a, \tau) &= \sup_{x, y \in \tau_{D(f)}(a)} \{(tf)(x) - (tf)(y)\} \\ &= \begin{cases} t \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\}, & t > 0 \\ t \inf_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\}, & t < 0 \end{cases} \\ &= \begin{cases} |t| \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\}, & t > 0 \\ |t| \sup_{x, y \in \tau_{D(f)}(a)} \{f(y) - f(x)\}, & t < 0 \end{cases} \\ &\leq 2|t|\varepsilon(a) \end{aligned}$$

Thus, $\text{slim}(tf, a, \tau, |t|\varepsilon) \neq \emptyset$.

($\Leftarrow$): Let $\text{slim}(tf, a, \tau, |t|\varepsilon) \neq \emptyset$. Then, $\Delta(tf, a, \tau) = \sup_{x, y \in \tau_{D(f)}(a)} \{(tf)(x) - (tf)(y)\} \leq 2|t|\varepsilon(a)$. Therefore,

$$\begin{aligned} \Delta(f, a, \tau) &= \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \\ &= \frac{|t|}{|t|} \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \\ &= \frac{1}{|t|} \sup_{x, y \in \tau_{D(f)}(a)} \{|t|f(x) - |t|f(y)\} \\ &= \begin{cases} \frac{1}{|t|} \sup_{x, y \in \tau_{D(f)}(a)} \{tf(x) - tf(y)\}, & t > 0 \\ \frac{1}{|t|} \sup_{x, y \in \tau_{D(f)}(a)} \{tf(y) - tf(x)\}, & t < 0 \end{cases} \\ &= \frac{1}{|t|} \sup_{x, y \in \tau_{D(f)}(a)} \{(tf)(x) - (tf)(y)\} \\ &\leq \frac{1}{|t|} 2|t|\varepsilon(a) \\ &= 2\varepsilon(a) \end{aligned}$$

Thus, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$.

Moreover, for $t < 0$,

$$\begin{aligned}\text{slim}(tf, a, \tau, |t|\varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{tf(x)\} - |t|\varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{tf(x)\} + |t|\varepsilon(a) \right] \\ &= \left[\sup_{x \in \tau_{D(f)}(a)} \{tf(x)\} + t\varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{tf(x)\} - t\varepsilon(a) \right] \\ &= \left[t \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right), t \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a) \right) \right] \\ &= t \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= t \text{slim}(f, a, \tau, \varepsilon)\end{aligned}$$

and for $t > 0$,

$$\begin{aligned}\text{slim}(tf, a, \tau, |t|\varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{tf(x)\} - t\varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{tf(x)\} + t\varepsilon(a) \right] \\ &= t \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= t \text{slim}(f, a, \tau, \varepsilon)\end{aligned}$$

Consequently, $\text{slim}(tf, a, \tau, |t|\varepsilon) = t \text{slim}(f, a, \tau, \varepsilon)$. $\square$

Corollary 9 ([3]). $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\text{slim}(-f, a, \tau, \varepsilon) \neq \emptyset$. Moreover,

$$\text{slim}(-f, a, \tau, \varepsilon) = -\text{slim}(f, a, \tau, \varepsilon)$$

Corollary 10 ([3]). Let $t > 0$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, a, \tau, t\varepsilon) \neq \emptyset$. Moreover, $\text{slim}(tf, a, \tau, t\varepsilon) = t \text{slim}(f, a, \tau, \varepsilon)$.

Theorem 31. Let $t \neq 0$. Then, $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, a, \tau^-, |t|\varepsilon) \neq \emptyset$. Moreover, $\text{slim}(tf, a, \tau^-, |t|\varepsilon) = t \text{slim}(f, a, \tau^-, \varepsilon)$.

Theorem 32. Let $t \neq 0$. Then, $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, a, \tau^+, |t|\varepsilon) \neq \emptyset$. Moreover, $\text{slim}(tf, a, \tau^+, |t|\varepsilon) = t \text{slim}(f, a, \tau^+, \varepsilon)$.

Theorem 33 ([4]). Let $t > 0$. Then, $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\overline{\text{slim}}(tf, a, \tau, t\varepsilon) \neq \emptyset$. Moreover, $\overline{\text{slim}}(tf, a, \tau, t\varepsilon) = t \overline{\text{slim}}(f, a, \tau, \varepsilon)$. Similarly, $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\underline{\text{slim}}(tf, a, \tau, t\varepsilon) \neq \emptyset$. In addition, $\underline{\text{slim}}(tf, a, \tau, t\varepsilon) = t \underline{\text{slim}}(f, a, \tau, \varepsilon)$.

Theorem 34. Let $t < 0$. Then, $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\underline{\text{slim}}(tf, a, \tau, |t|\varepsilon) \neq \emptyset$. Moreover, $\underline{\text{slim}}(tf, a, \tau, |t|\varepsilon) = \overline{\text{slim}}(f, a, \tau, \varepsilon)$. Similarly, $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\overline{\text{slim}}(tf, a, \tau, |t|\varepsilon) \neq \emptyset$. In addition, $\overline{\text{slim}}(tf, a, \tau, |t|\varepsilon) = \underline{\text{slim}}(f, a, \tau, \varepsilon)$.

The proofs of Theorems 31–34 are similar to the proof of Theorem 30.

Example 7. For the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 1$ and $g(x) = -2x - 2$, respectively, $\varepsilon(1) = 3$, and $\tau_2(1) = [-1, 1) \cup (1, 3]$,

$$\text{slim}(g, 1, \tau_2, | - 2|\varepsilon) = \text{slim}(-2f, 1, \tau_2, 2\varepsilon) = [-6, -2] = -2[1, 3] = -2\text{slim}(f, 1, \tau_2, \varepsilon)$$

Theorem 35 (Sum Rule). If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, then $\text{slim}(f + g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover,

$$\text{slim}(f, a, \tau, \alpha) + \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(f + g, a, \kappa, \varepsilon)$$

Proof. Let $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$. Then, $\Delta(f, a, \tau) \leq 2\alpha(a)$ and $\Delta(g, a, \lambda) \leq 2\beta(a)$. Therefore,

$$\begin{aligned} \Delta(f + g, a, \kappa) &= \sup_{x, y \in \kappa_{D(f)}(a)} \{(f + g)(x) - (f + g)(y)\} \\ &\leq \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} + \sup_{x, y \in \lambda_{D(f)}(a)} \{g(x) - g(y)\} \\ &\leq 2\varepsilon(a) \end{aligned}$$

Thus, $\text{slim}(f + g, a, \kappa, \varepsilon) \neq \emptyset$. Moreover, for $S = \text{slim}(f, a, \tau, \alpha) + \text{slim}(g, a, \lambda, \beta)$,

$$\begin{aligned} S &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right] \\ &\quad + \left[\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a), \inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right] \\ &\subseteq \left[\sup_{x \in \kappa_{D(f)}(a)} \{(f + g)(x)\} - (\alpha(a) + \beta(a)), \inf_{x \in \kappa_{D(f)}(a)} \{(f + g)(x)\} + (\alpha(a) + \beta(a)) \right] \\ &\subseteq \left[\sup_{x \in \kappa_{D(f)}(a)} \{(f + g)(x)\} - \varepsilon(a), \inf_{x \in \kappa_{D(f)}(a)} \{(f + g)(x)\} + \varepsilon(a) \right] \\ &= \text{slim}(f + g, a, \kappa, \varepsilon) \end{aligned}$$

□

Corollary 11 ([3]). If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \tau, \beta) \neq \emptyset$, then $\text{slim}(f + g, a, \tau, \alpha + \beta) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \alpha) + \text{slim}(g, a, \tau, \beta) \subseteq \text{slim}(f + g, a, \tau, \alpha + \beta)$.

Theorem 36. If $\text{slim}(f, a, \tau^-, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda^-, \beta) \neq \emptyset$, then $\text{slim}(f + g, a, \kappa^-, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}^-(a) \subseteq \tau_{D(f)}^-(a) \cap \lambda_{D(f)}^-(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau^-, \alpha) + \text{slim}(g, a, \lambda^-, \beta) \subseteq \text{slim}(f + g, a, \kappa^-, \varepsilon)$.

Theorem 37. If $\text{slim}(f, a, \tau^+, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda^+, \beta) \neq \emptyset$, then $\text{slim}(f + g, a, \kappa^+, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}^+(a) \subseteq \tau_{D(f)}^+(a) \cap \lambda_{D(f)}^+(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau^+, \alpha) + \text{slim}(g, a, \lambda^+, \beta) \subseteq \text{slim}(f + g, a, \kappa^+, \varepsilon)$.

Theorem 38. If $\overline{\text{slim}}(f, a, \tau, \alpha) \neq \emptyset$ and $\overline{\text{slim}}(g, a, \lambda, \beta) \neq \emptyset$, then $\overline{\text{slim}}(f + g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\overline{\text{slim}}(f, a, \tau, \alpha) + \overline{\text{slim}}(g, a, \lambda, \beta) \subseteq \overline{\text{slim}}(f + g, a, \kappa, \varepsilon)$.

Theorem 39. If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, then $\text{slim}(f + g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau, \alpha) + \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(f + g, a, \kappa, \varepsilon)$.

The proofs of Theorems 36–39 are similar to the proof of Theorem 35.

Corollary 12 (Difference Rule). If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, then $\text{slim}(f - g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau, \alpha) - \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(f - g, a, \kappa, \varepsilon)$.

Corollary 13. If $\text{slim}(f, a, \tau^-, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda^-, \beta) \neq \emptyset$, then $\text{slim}(f - g, a, \kappa^-, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}^-(a) \subseteq \tau_{D(f)}^-(a) \cap \lambda_{D(f)}^-(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau^-, \alpha) - \text{slim}(g, a, \lambda^-, \beta) \subseteq \text{slim}(f - g, a, \kappa^-, \varepsilon)$.

Corollary 14. If $\text{slim}(f, a, \tau^+, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda^+, \beta) \neq \emptyset$, then $\text{slim}(f - g, a, \kappa^+, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}^+(a) \subseteq \tau_{D(f)}^+(a) \cap \lambda_{D(f)}^+(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau^+, \alpha) - \text{slim}(g, a, \lambda^+, \beta) \subseteq \text{slim}(f - g, a, \kappa^+, \varepsilon)$.

Corollary 15. If $\overline{\text{slim}}(f, a, \tau, \alpha) \neq \emptyset$ and $\overline{\text{slim}}(g, a, \lambda, \beta) \neq \emptyset$, then $\overline{\text{slim}}(f - g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \overline{\kappa_{D(f)}}(a) \subseteq \overline{\tau_{D(f)}}(a) \cap \overline{\lambda_{D(f)}}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\overline{\text{slim}}(f, a, \tau, \alpha) - \overline{\text{slim}}(g, a, \lambda, \beta) \subseteq \overline{\text{slim}}(f - g, a, \kappa, \varepsilon)$.

Corollary 16. If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, then $\text{slim}(f - g, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\text{slim}(f, a, \tau, \alpha) - \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(f - g, a, \kappa, \varepsilon)$.

Example 8. For the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$ and $g(x) = x^2$, respectively, $\alpha(1) = 3$, $\beta(1) = 7$, $\varepsilon(1) = 11$, $\tau_2(1) = [-1, 1] \cup (1, 3]$, $\lambda_{\frac{5}{2}}(1) = [-\frac{3}{2}, 1] \cup (1, \frac{7}{2}]$, and $\kappa_1(1) = [0, 1] \cup (1, 2]$,

$$\text{slim}(f, 1, \tau_2, \alpha) + \text{slim}(g, 1, \lambda_{\frac{5}{2}}, \beta) = \left[\frac{33}{4}, 12 \right] \subseteq [-2, 14] = \text{slim}(f + g, 1, \kappa_1, \varepsilon)$$

and

$$\text{slim}(f, 1, \tau_2, \alpha) - \text{slim}(g, 1, \lambda_{\frac{5}{2}}, \beta) = \left[-4, -\frac{1}{4} \right] \subseteq \left[-\frac{31}{4}, 12 \right] = \text{slim}(f - g, 1, \kappa_1, \varepsilon)$$

Here, $\text{slim}(f, 1, \tau_2, \alpha) = [3, 5]$ and $\text{slim}(g, 1, \lambda_{\frac{5}{2}}, \beta) = \left[\frac{21}{4}, 7 \right]$.

Lemma 1. Let $B, C \subseteq A$, $D \subseteq B \cap C$, $f(B), g(C) \subseteq \mathbb{R}^{\geq 0}$, and $k, l \in \mathbb{R}^{\geq 0}$. Then,

$$\sup_D fg - \left(\sup_B f + k \right) \left(\sup_C g + l \right) \leq \min \left\{ \left(\sup_B f - k \right) \left(\sup_C g - l \right), \left(\sup_B f - k \right) \left(\inf_C g + l \right), \right. \\ \left. \left(\inf_B f + k \right) \left(\sup_C g - l \right), \left(\inf_B f + k \right) \left(\inf_C g + l \right) \right\}$$

and

$$\inf_D fg + \left(\sup_B f + k\right) \left(\sup_C g + l\right) \geq \max \left\{ \left(\sup_B f - k\right) \left(\sup_C g - l\right), \left(\sup_B f - k\right) \left(\inf_C g + l\right), \right. \\ \left. \left(\inf_B f + k\right) \left(\sup_C g - l\right), \left(\inf_B f + k\right) \left(\inf_C g + l\right) \right\}$$

Proof. Let $B, C \subseteq A$, $D \subseteq B \cap C$, $f(B), g(C) \subseteq \mathbb{R}^{\geq 0}$, and $k, l \in \mathbb{R}^{\geq 0}$. Since $f(B), g(C) \subseteq \mathbb{R}^{\geq 0}$, then $0 \leq \inf_B f \leq \sup_B f$, $0 \leq \inf_C g \leq \sup_C g$, and $0 \leq \inf_B f \inf_C g \leq \inf_D f \inf_D g \leq \inf_D fg \leq \sup_D fg \leq \sup_D f \sup_D g \leq \sup_B f \sup_C g$. Assume that

$$\left(\inf_B f + k\right) \left(\inf_C g + l\right) = \min \left\{ \left(\sup_B f - k\right) \left(\sup_C g - l\right), \left(\sup_B f - k\right) \left(\inf_C g + l\right), \right. \\ \left. \left(\inf_B f + k\right) \left(\sup_C g - l\right), \left(\inf_B f + k\right) \left(\inf_C g + l\right) \right\}$$

and

$$\left(\sup_B f - k\right) \left(\sup_C g - l\right) = \max \left\{ \left(\sup_B f - k\right) \left(\sup_C g - l\right), \left(\sup_B f - k\right) \left(\inf_C g + l\right), \right. \\ \left. \left(\inf_B f + k\right) \left(\sup_C g - l\right), \left(\inf_B f + k\right) \left(\inf_C g + l\right) \right\}$$

Then, for $S_1 = \sup_D fg - \left(\sup_B f + k\right) \left(\sup_C g + l\right) - \left(\inf_B f + k\right) \left(\inf_C g + l\right)$,

$$\begin{aligned} S_1 &= \sup_D fg - \sup_B f \sup_C g - l \sup_B f - k \sup_C g - kl - \inf_B f \inf_C g - l \inf_B f - k \inf_C g - kl \\ &\leq \sup_B f \sup_C g - \sup_B f \sup_C g - \left(l \sup_D f + k \sup_D g + \inf_B f \inf_C g + l \inf_B f + k \inf_C g + 2kl \right) \\ &\leq - \left(l \sup_D f + k \sup_D g + \inf_B f \inf_C g + l \inf_B f + k \inf_C g + 2kl \right) \\ &\leq 0 \end{aligned}$$

and for $S_2 = \left(\sup_B f - k\right) \left(\sup_C g - l\right) - \left(\inf_D fg + \left(\sup_B f + k\right) \left(\sup_C g + l\right)\right)$,

$$\begin{aligned} S_2 &= \sup_B f \sup_C g - l \sup_B f - k \sup_C g + kl - \inf_D fg - \sup_B f \sup_C g - l \sup_B f - k \sup_C g - kl \\ &\leq - \left(l \sup_B f + k \sup_C g + \inf_D fg + l \sup_D f + k \sup_D g \right) \\ &\leq 0 \end{aligned}$$

The inequalities in the lemma for the other three assumptions can be similarly proven. $\square$

Lemma 1 provides the groundwork for Theorem 40, Proposition 1, and Corollary 17.

Theorem 40 (Product Rule for nonnegative-valued functions). *If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^{\geq 0}$, then $\text{slim}(fg, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and*

$$\left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right) \leq \varepsilon(a)$$

Moreover,

$$\text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(fg, a, \kappa, \varepsilon)$$

Proof. Let $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^{\geq 0}$. Then, $\Delta(f, a, \tau) \leq 2\alpha(a)$ and $\Delta(g, a, \lambda) \leq 2\beta(a)$. Assume that

$$P = \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right)$$

$$Q = \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right)$$

$$R = \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right)$$

and

$$T = \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right)$$

Since, for all $x, y \in \kappa_{D(f)}(a)$,

$$\begin{aligned} (fg)(x) - (fg)(y) &= f(x)(g(x) - g(y)) + g(y)(f(x) - f(y)) \\ &\leq \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} \sup_{x, y \in \lambda_{D(f)}(a)} \{g(x) - g(y)\} \\ &\quad + \sup_{y \in \lambda_{D(f)}(a)} \{g(y)\} \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \\ &\leq 2\beta(a) \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + 2\alpha(a) \sup_{y \in \lambda_{D(f)}(a)} \{g(y)\} \\ &\leq 2 \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right) \\ &\leq 2\varepsilon(a) \end{aligned} \tag{1}$$

then

$$\Delta(fg, a, \kappa) = \sup_{x, y \in \kappa_{D(f)}(a)} \{(fg)(x) - (fg)(y)\} \leq 2\varepsilon(a)$$

Therefore, $\text{slim}(fg, a, \kappa, \varepsilon) \neq \emptyset$. As $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^{\geq 0}$ and $\kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$, from Lemma 1,

$$\sup_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} - \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right) \leq \min\{P, Q, R, T\}$$

and

$$\inf_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} + \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right) \geq \max\{P, Q, R, T\}$$

Moreover, for $S = \text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta)$,

$$\begin{aligned} S &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right] \left[\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a), \inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right] \\ &= [\min\{P, Q, R, T\}, \max\{P, Q, R, T\}] \\ &\subseteq \left[\sup_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} - \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right), \right. \\ &\quad \left. \inf_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} + \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right) \right] \\ &= \left[\sup_{x \in \kappa_{D(f)}(a)} \{(fg)(x)\} - \varepsilon(a), \inf_{x \in \kappa_{D(f)}(a)} \{(fg)(x)\} + \varepsilon(a) \right] \\ &= \text{slim}(fg, a, \kappa, \varepsilon) \end{aligned}$$

□

Note 5. As can be observed from the third line of Inequality 1, for all $x, y \in \kappa_{D(f)}(a)$,

$$(fg)(x) - (fg)(y) \leq 2 \left(\beta(a) \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \sup_{y \in \lambda_{D(f)}(a)} \{g(y)\} \right)$$

Hence, $\Delta(fg, a, \kappa) \leq 2 \left(\beta(a) \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \sup_{y \in \lambda_{D(f)}(a)} \{g(y)\} \right)$. Thus, $\text{slim}(fg, a, \kappa, \varepsilon^*) \neq \emptyset$ such that $\beta(a) \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \sup_{y \in \lambda_{D(f)}(a)} \{g(y)\} \leq \varepsilon^*(a)$. However, the inclusion

$$\text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(fg, a, \kappa, \varepsilon^*)$$

is not always valid. Moreover, it is worth studying the conditions under which the equality $\text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta) = \text{slim}(fg, a, \kappa, \varepsilon)$ is valid.

Proposition 1 (Quotient Rule for nonnegative-valued functions). If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, $0 \notin \text{slim}(g, a, \lambda, \beta)$, $f(\tau_{D(f)}(a)) \subseteq \mathbb{R}^{\geq 0}$, and $g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^+$, then $\text{slim}\left(\frac{f}{g}, a, \kappa, \varepsilon\right) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \cap \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and

$$\left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \left\{ \frac{1}{g(x)} \right\} + \beta(a) \right) \leq \varepsilon(a)$$

Moreover,

$$\text{slim}(f, a, \tau, \alpha) \text{slim}\left(\frac{1}{g}, a, \lambda, \beta\right) \subseteq \text{slim}\left(\frac{f}{g}, a, \kappa, \varepsilon\right)$$

Corollary 17 (Power Rule for nonnegative-valued functions). *If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$ and $f(\tau_{D(f)}(a)) \subseteq \mathbb{R}^{\geq 0}$, then $\text{slim}(f^n, a, \tau, \beta^{n-1}) \neq \emptyset$ such that $\beta^1 = \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right)^2$ and $\beta^{n-1} = \left(\sup_{x \in \tau_{D(f)}(a)} \{f^{n-1}(x)\} + \beta^{n-2}(a) \right) \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right)$, for $n \in \mathbb{N}_3$, the set of all natural numbers greater than 2. Moreover, $\text{slim}^n(f, a, \tau, \alpha) \subseteq \text{slim}(f^n, a, \tau, \beta^{n-1})$. Here, $f^n(x) = (f(x))^n$, for all $x \in A$, and $\text{slim}^n(f, a, \tau, \alpha) = (\text{slim}(f, a, \tau, \alpha))^n$.*

Example 9. For the functions $f, g : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ defined by $f(x) = x$ and $g(x) = x^2$, respectively, $\alpha(2) = 3$, $\beta(2) = 5$, $\tau_2(2) = [0, 2) \cup (2, 4]$, and $\lambda_1(2) = [1, 2) \cup (2, 3]$, $f(\tau_2(2) \cap \mathbb{R}^{\geq 0}), g(\lambda_1(2) \cap \mathbb{R}^{\geq 0}) \subseteq \mathbb{R}^{\geq 0}$. Moreover,

$$\text{slim}(f, 2, \tau_2, \alpha) \text{slim}(g, 2, \lambda_1, \beta) = [1, 3][4, 6] = [4, 18] \subseteq \left[-\frac{675}{8}, \frac{827}{8} \right] = \text{slim}(fg, 2, \kappa_{\frac{1}{2}}, \varepsilon)$$

where $\kappa_{\frac{1}{2}}(2) = [\frac{3}{2}, 2) \cup (2, \frac{5}{2}]$ and $\left(\sup_{x \in \tau_2(2) \cap \mathbb{R}^{\geq 0}} \{x\} + \alpha(2) \right) \left(\sup_{x \in \lambda_1(2) \cap \mathbb{R}^{\geq 0}} \{x^2\} + \beta(2) \right) = 98 \leq 100 = \varepsilon(2)$.

Lemma 2. Let $B, C \subseteq A$, $D \subseteq B \cap C$, $f(B), g(C) \subseteq \mathbb{R}^-$, and $k, l \in \mathbb{R}^{\geq 0}$. Then,

$$\sup_D fg - \left(\inf_B f - k \right) \left(\inf_C g - l \right) \leq \min \left\{ \left(\sup_B f - k \right) \left(\sup_C g - l \right), \left(\sup_B f - k \right) \left(\inf_C g + l \right), \right. \\ \left. \left(\inf_B f + k \right) \left(\sup_C g - l \right), \left(\inf_B f + k \right) \left(\inf_C g + l \right) \right\}$$

and

$$\inf_D fg + \left(\inf_B f - k \right) \left(\inf_C g - l \right) \geq \max \left\{ \left(\sup_B f - k \right) \left(\sup_C g - l \right), \left(\sup_B f - k \right) \left(\inf_C g + l \right), \right. \\ \left. \left(\inf_B f + k \right) \left(\sup_C g - l \right), \left(\inf_B f + k \right) \left(\inf_C g + l \right) \right\}$$

The proof of Lemma 2 is similar to the proof of Lemma 1. Moreover, Lemma 2 forms the basis for proving Theorem 41 and Proposition 2.

Theorem 41 (Product Rule for negative-valued functions). *If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^-$, then $\text{slim}(fg, a, \kappa, \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and $\left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right) \leq \varepsilon(a)$. Moreover,*

$$\text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta) \subseteq \text{slim}(fg, a, \kappa, \varepsilon)$$

Proof. Let $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^-$. Then, $\Delta(f, a, \tau) \leq 2\alpha(a)$ and $\Delta(g, a, \lambda) \leq 2\beta(a)$. Assume that

$$P = \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right)$$

$$Q = \left(\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right)$$

$$R = \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right)$$

and

$$T = \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right)$$

Since, for all $x, y \in \kappa_{D(f)}(a)$,

$$\begin{aligned} (fg)(x) - (fg)(y) &= f(x)g(x) - f(y)g(y) \\ &= (-f(x))(g(y) - g(x)) + (-g(y))(f(y) - f(x)) \\ &\leq \sup_{x \in \tau_{D(f)}(a)} \{-f(x)\} \sup_{x, y \in \lambda_{D(f)}(a)} \{g(y) - g(x)\} \\ &\quad + \sup_{y \in \lambda_{D(f)}(a)} \{-g(y)\} \sup_{x, y \in \tau_{D(f)}(a)} \{f(y) - f(x)\} \\ &\leq 2\beta(a) \sup_{x \in \tau_{D(f)}(a)} \{-f(x)\} + 2\alpha(a) \sup_{y \in \lambda_{D(f)}(a)} \{-g(y)\} \\ &\leq 2 \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right) \\ &\leq 2\varepsilon(a) \end{aligned}$$

then

$$\Delta(fg, a, \kappa) = \sup_{x, y \in \kappa_{D(f)}(a)} \{(fg)(x) - (fg)(y)\} \leq 2\varepsilon(a)$$

Therefore, $\text{slim}(fg, a, \kappa, \varepsilon) \neq \emptyset$. As $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^-$ and $\kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$, from Lemma 2,

$$\sup_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} - \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right) \leq \min\{P, Q, R, T\}$$

and

$$\inf_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} + \left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a) \right) \geq \max\{P, Q, R, T\}$$

Moreover, for $S = \text{slim}(f, a, \tau, \alpha) \text{slim}(g, a, \lambda, \beta)$,

$$\begin{aligned}
 S &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \alpha(a) \right] \left[\sup_{x \in \lambda_{D(f)}(a)} \{g(x)\} - \beta(a), \inf_{x \in \lambda_{D(f)}(a)} \{g(x)\} + \beta(a) \right] \\
 &= [\min\{P, Q, R, T\}, \max\{P, Q, R, T\}] \\
 &\subseteq \left[\sup_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} - \left(\sup_{x \in \tau_{D(f)}(a)} \{-f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{-g(x)\} + \beta(a) \right), \right. \\
 &\quad \left. \inf_{x \in \kappa_{D(f)}(a)} \{f(x)g(x)\} + \left(\sup_{x \in \tau_{D(f)}(a)} \{-f(x)\} + \alpha(a) \right) \left(\sup_{x \in \lambda_{D(f)}(a)} \{-g(x)\} + \beta(a) \right) \right] \\
 &= \left[\sup_{x \in \kappa_{D(f)}(a)} \{(fg)(x)\} - \varepsilon(a), \inf_{x \in \kappa_{D(f)}(a)} \{(fg)(x)\} + \varepsilon(a) \right] \\
 &= \text{slim}(fg, a, \kappa, \varepsilon) \quad \square
 \end{aligned}$$

Proposition 2 (Quotient Rule for negative-valued functions). If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, a, \lambda, \beta) \neq \emptyset$, $0 \notin \text{slim}(g, a, \tau, \beta)$, and $f(\tau_{D(f)}(a)), g(\lambda_{D(f)}(a)) \subseteq \mathbb{R}^-$, then $\text{slim}\left(\frac{f}{g}, a, \kappa, \varepsilon\right) \neq \emptyset$ such that $\emptyset \neq \kappa_{D(f)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(f)}(a)$ and

$$\left(\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} - \alpha(a) \right) \left(\inf_{x \in \lambda_{D(f)}(a)} \left\{ \frac{1}{g(x)} \right\} - \beta(a) \right) \leq \varepsilon(a)$$

Moreover,

$$\text{slim}(f, a, \tau, \alpha) \text{slim}\left(\frac{1}{g}, a, \lambda, \beta\right) \subseteq \text{slim}\left(\frac{f}{g}, a, \kappa, \varepsilon\right)$$

Example 10. For the functions $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $f(x) = -x$ and $g(x) = -x^2$, respectively, $\alpha(2) = 3$, $\beta(2) = 5$, $\tau_2(2) = [0, 2) \cup (2, 4]$, and $\lambda_1(2) = [1, 2) \cup (2, 3]$, $f(\tau_2(2) \cap \mathbb{R}^+), g(\lambda_1(2) \cap \mathbb{R}^+) \subseteq \mathbb{R}^-$. Moreover,

$$\text{slim}(f, 2, \tau_2, \alpha) \text{slim}(g, 2, \lambda_1, \beta) = [-3, -1][-6, -4] = [4, 18] \subseteq \left[-\frac{675}{8}, \frac{827}{8}\right] = \text{slim}\left(fg, 2, \kappa_{\frac{1}{2}}, \varepsilon\right)$$

where $\kappa_{\frac{1}{2}}(2) = [\frac{3}{2}, 2) \cup (2, \frac{5}{2}]$ and $\left(\inf_{x \in \tau_2(2) \cap \mathbb{R}^+} \{-x\} - \alpha(2) \right) \left(\inf_{x \in \lambda_1(2) \cap \mathbb{R}^+} \{-x^2\} - \beta(2) \right) = 98 \leq 100 = \varepsilon(2)$.

Theorem 42 (Restriction Rule). Let $\emptyset \neq B \subseteq A$ and $\tau_{D(f|_B)}(a) \neq \emptyset$. If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, then $\text{slim}(f|_B, a, \tau, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \varepsilon) \subseteq \text{slim}(f|_B, a, \tau, \varepsilon)$.

Proof. Let $\emptyset \neq B \subseteq A$, $\tau_{D(f|_B)}(a) \neq \emptyset$, and $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Then, $\Delta(f, a, \tau) = \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \leq 2\varepsilon$. Therefore,

$$\begin{aligned}
 \Delta(f|_B, a, \tau) &= \sup_{x, y \in \tau_{D(f|_B)}(a)} \{f|_B(x) - f|_B(y)\} \\
 &= \sup_{x, y \in \tau_{D(f|_B)}(a)} \{f(x) - f(y)\} \\
 &\leq \sup_{x, y \in \tau_{D(f)}(a)} \{f(x) - f(y)\} \\
 &\leq 2\varepsilon
 \end{aligned}$$

Thus, $\text{slim}(f|_B, a, \tau, \varepsilon) \neq \emptyset$. Moreover, since

$$\sup_{x \in \tau_{D(f|_B)}(a)} \{f|_B(x)\} = \sup_{x \in \tau_{D(f)}(a)} \{f(x)\} \leq \sup_{x \in \tau_{D(f)}(a)} \{f(x)\}$$

and

$$\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} \leq \inf_{x \in \tau_{D(f|_B)}(a)} \{f(x)\} = \inf_{x \in \tau_{D(f|_B)}(a)} \{f|_B(x)\}$$

then

$$\begin{aligned} \text{slim}(f, a, \tau, \varepsilon) &= \left[\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &\subseteq \left[\sup_{x \in \tau_{D(f|_B)}(a)} \{f(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f|_B)}(a)} \{f(x)\} + \varepsilon(a) \right] \\ &= \left[\sup_{x \in \tau_{D(f|_B)}(a)} \{f|_B(x)\} - \varepsilon(a), \inf_{x \in \tau_{D(f|_B)}(a)} \{f|_B(x)\} + \varepsilon(a) \right] \\ &= \text{slim}(f|_B, a, \tau, \varepsilon) \end{aligned}$$

□

Example 11. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x + 1$, $\varepsilon(3) = 5$, and $\tau_2(3) = [1, 3) \cup (3, 5]$, $\text{slim}(f, 3, \tau_2, \varepsilon) = [6, 8]$ and $\text{slim}(f|_{(3,4)}, 3, \tau_2, \varepsilon) = [4, 12]$. Therefore, $\text{slim}(f, 3, \tau_2, \varepsilon) \subseteq \text{slim}(f|_{(3,4)}, 3, \tau_2, \varepsilon)$.

Theorem 43 (Composition Rule). Let $g : B \rightarrow \mathbb{R}$ be a function, $b \in \mathbb{R}$, $f(A) \subseteq B$, and $\beta(b) \leq \alpha(a)$. If $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, b, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)) \subseteq \lambda_{D(g)}(b)$, then $\text{slim}(g \circ f, a, \tau, \alpha) \neq \emptyset$. Moreover, $\text{slim}(g, b, \lambda, \beta) \subseteq \text{slim}(g \circ f, a, \tau, \alpha)$.

Proof. Let $g : B \rightarrow \mathbb{R}$ be a function, $b \in \mathbb{R}$, $f(A) \subseteq B$, $\beta(b) \leq \alpha(a)$, $\text{slim}(f, a, \tau, \alpha) \neq \emptyset$, $\text{slim}(g, b, \lambda, \beta) \neq \emptyset$, and $f(\tau_{D(f)}(a)) \subseteq \lambda_{D(g)}(b)$. Then, $\Delta(f, a, \tau) \leq 2\alpha(a)$ and $\Delta(g, b, \lambda) \leq 2\beta(b)$. Hence,

$$\begin{aligned} \Delta(g \circ f, a, \tau) &= \sup_{x, y \in \tau_{D(g \circ f)}(a)} \{(g \circ f)(x) - (g \circ f)(y)\} \\ &= \sup_{x, y \in \tau_{D(f)}(a)} \{g(f(x)) - g(f(y))\} \\ &= \sup_{z, t \in f(\tau_{D(f)}(a))} \{g(z) - g(t)\} \\ &\leq \sup_{z, t \in \lambda_{D(g)}(b)} \{g(z) - g(t)\} \\ &\leq 2\beta(b) \\ &\leq 2\alpha(a) \end{aligned}$$

Therefore, $\text{slim}(g \circ f, a, \tau, \alpha) \neq \emptyset$. Moreover, since $f(\tau_{D(f)}(a)) \subseteq \lambda_{D(g)}(b)$, then

$$\sup_{x \in f(\tau_{D(f)}(a))} \{g(x)\} \leq \sup_{x \in \lambda_{D(g)}(b)} \{g(x)\} \quad \text{and} \quad \inf_{x \in \lambda_{D(g)}(b)} \{g(x)\} \leq \inf_{x \in f(\tau_{D(f)}(a))} \{g(x)\}$$

Therefore,

$$\begin{aligned} \text{slim}(g, b, \lambda, \beta) &= \left[\sup_{y \in \lambda_{D(g)}(b)} \{g(y)\} - \beta(b), \inf_{y \in \lambda_{D(g)}(b)} \{g(y)\} + \beta(b) \right] \\ &\subseteq \left[\sup_{y \in f(\tau_{D(f)}(a))} \{g(y)\} - \beta(b), \inf_{y \in f(\tau_{D(f)}(a))} \{g(y)\} + \beta(b) \right] \\ &\subseteq \left[\sup_{x \in \tau_{D(g \circ f)}(a)} \{(g \circ f)(x)\} - \alpha(a), \inf_{x \in \tau_{D(g \circ f)}(a)} \{(g \circ f)(x)\} + \alpha(a) \right] \\ &= \text{slim}(g \circ f, a, \tau, \alpha) \end{aligned}$$

□

Corollary 18. Let $g : B \rightarrow \mathbb{R}$ be a function, $b \in \mathbb{R}$, $f(A) \subseteq B$, and $\varepsilon(b) \leq \varepsilon(a)$. If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, $\text{slim}(g, b, \tau, \varepsilon) \neq \emptyset$, and $f(\tau_{D(f)}(a)) \subseteq \tau_{D(g)}(b)$, then $\text{slim}(g \circ f, a, \tau, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(g, b, \tau, \varepsilon) \subseteq \text{slim}(g \circ f, a, \tau, \varepsilon)$.

Example 12. For the functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 5 - x$ and $g(x) = x + 1$, respectively, $a = 2$, $b = 3$, $\alpha(2) = 6$, and $\beta(3) = 5$, $f(\mathbb{R}^+) = (-\infty, 5) \subseteq \mathbb{R}$, $\beta(3) \leq \alpha(2)$, and $f(\tau_1(2) \cap \mathbb{R}^+) = f([1, 2) \cup (2, 3]) = [2, 3) \cup (3, 4] \subseteq [-1, 3) \cup (3, 7] = \tau_4(3) \cap \mathbb{R}$. Then, $\text{slim}(g, 3, \tau_4, \beta) = [3, 5] \subseteq [-1, 9] = \text{slim}(g \circ f, 2, \tau_1, \alpha)$.

Proposition 3 (Identity Function). For the identity function $Id_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Id_{\mathbb{R}}(x) = x$ and $\delta(a) \leq \varepsilon(a)$, $\text{slim}(Id_{\mathbb{R}}, a, \tau_{\delta}, \varepsilon) \neq \emptyset$. Moreover,

$$\text{slim}(Id_{\mathbb{R}}, a, \tau_{\delta}, \varepsilon) = [a - (\varepsilon(a) - \delta(a)), a + (\varepsilon(a) - \delta(a))]$$

Note 6. In Proposition 3, if $\delta(a) = \varepsilon(a)$, then $\text{slim}(Id_{\mathbb{R}}, a, \tau_{\delta}, \varepsilon) = [a, a] = a$. That is, the $(\tau_{\delta}, \varepsilon)$ -soft limit of $Id_{\mathbb{R}}$ at a becomes the classical limit of $Id_{\mathbb{R}}$ at a . Moreover, if $\delta(a) > \varepsilon(a)$, then $\text{slim}(Id_{\mathbb{R}}, a, \tau_{\delta}, \varepsilon) = \emptyset$.

Theorem 44. Let $a \in A \cap B$, $g : B \rightarrow \mathbb{R}$ be a function, $\tau(a) \subseteq A \cap B$, and $f(x) = g(x)$, for all $x \in \tau(a)$. Then, $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ iff $\text{slim}(g, a, \tau, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \varepsilon) = \text{slim}(g, a, \tau, \varepsilon)$.

Proof. Let $a \in A \cap B$, $g : B \rightarrow \mathbb{R}$ be a function, $\tau(a) \subseteq A \cap B$, and $f(x) = g(x)$, for all $x \in \tau(a)$.

($\Rightarrow$): Assume that $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow |f(x) - L| \leq \varepsilon(a)$$

Since $\tau_{D(f)}(a) = \tau_{D(g)}(a) = \tau(a)$, then

$$x \in \tau_{D(g)}(a) \Rightarrow |g(x) - L| = |f(x) - L| \leq \varepsilon(a)$$

Hence, $\text{slim}(g, a, \tau, \varepsilon) \neq \emptyset$.

($\Leftarrow$): The proof is similar.

Moreover, the equality $\text{slim}(f, a, \tau, \varepsilon) = \text{slim}(g, a, \tau, \varepsilon)$ can be observed from the above proof. □

Theorem 45 (Comparison Theorem). *If $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, $\text{slim}(g, a, \tau, \varepsilon) \neq \emptyset$, and $f(x) \leq g(x)$, for all $x \in \tau_{D(f)}(a) = \tau_{D(g)}(a)$, then*

$$\text{slim}(f, a, \tau, \varepsilon) \lesssim \text{slim}(g, a, \tau, \varepsilon)$$

Proof. Let $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$, $\text{slim}(g, a, \tau, \varepsilon) \neq \emptyset$, and $f(x) \leq g(x)$, for all $x \in \tau_{D(f)}(a) = \tau_{D(g)}(a)$. Then,

$$\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} \leq \sup_{x \in \tau_{D(g)}(a)} \{g(x)\} \quad \text{and} \quad \inf_{x \in \tau_{D(f)}(a)} \{f(x)\} \leq \inf_{x \in \tau_{D(g)}(a)} \{g(x)\}$$

Therefore,

$$\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} - \varepsilon(a) \leq \sup_{x \in \tau_{D(g)}(a)} \{g(x)\} - \varepsilon(a)$$

and

$$\inf_{x \in \tau_{D(f)}(a)} \{f(x)\} + \varepsilon(a) \leq \inf_{x \in \tau_{D(g)}(a)} \{g(x)\} + \varepsilon(a)$$

Hence,

$$\text{slim}(f, a, \tau, \varepsilon) \lesssim \text{slim}(g, a, \tau, \varepsilon)$$

□

Example 13. For the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x - 3$ and $g(x) = 3x + 2$, respectively, and $\varepsilon(3) = 7$, $f(x) \leq g(x)$, for all $x \in \tau_2(3) \cap \mathbb{R} = ([1, 3] \cup (3, 5]) \cap \mathbb{R}$. Since $\text{slim}(f, 3, \tau_2, \varepsilon) = [0, 6]$ and $\text{slim}(g, 3, \tau_2, \varepsilon) = [10, 12]$, then $\text{slim}(f, 3, \tau_2, \varepsilon) \lesssim \text{slim}(g, 3, \tau_2, \varepsilon)$.

Theorem 46 (Sandwich/Squeeze Theorem). *Let $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$ and $\text{slim}(g, a, \tau, \varepsilon) \neq \emptyset$. If $f(x) \leq h(x) \leq g(x)$, for all $x \in \tau_{D(f)}(a) = \tau_{D(g)}(a) = \tau_{D(h)}(a)$, and $\text{slim}(f, a, \tau, \varepsilon) = \text{slim}(g, a, \tau, \varepsilon)$, then $\text{slim}(h, a, \tau, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \varepsilon) = \text{slim}(h, a, \tau, \varepsilon) = \text{slim}(g, a, \tau, \varepsilon)$.*

The proof follows from Theorem 45.

Example 14. For the functions $f, g, h : [-1, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^6$, $g(x) = x^2$, and $h(x) = x^4$, respectively, $a = 0$, and $\varepsilon(0) = 2$, $f(x) \leq h(x) \leq g(x)$, for all $x \in \tau_1(0) \cap [-1, 1] = ([-1, 0] \cup (0, 1]) \cap [-1, 1]$. Since

$$\Delta(f, 0, \tau_1) = \sup_{x, y \in \tau_1(0) \cap [-1, 1]} \{x^6 - y^6\} = 1 \leq 4 = 2\varepsilon(0)$$

and

$$\Delta(g, 0, \tau_1) = \sup_{x, y \in \tau_1(0) \cap [-1, 1]} \{x^2 - y^2\} = 1 \leq 4 = 2\varepsilon(0)$$

then $\text{slim}(f, 0, \tau_1, \varepsilon) \neq \emptyset$ and $\text{slim}(g, 0, \tau_1, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, 0, \tau_1, \varepsilon) = [-1, 2] = \text{slim}(g, 0, \tau_1, \varepsilon)$. In addition,

$$\Delta(h, 0, \tau_1) = \sup_{x, y \in \tau_1(0) \cap [-1, 1]} \{x^4 - y^4\} = 1 \leq 4 = 2\varepsilon(0)$$

and

$$\text{slim}(h, 0, \tau_1, \varepsilon) = [-1, 2]$$

Consequently, $\text{slim}(f, 0, \tau_1, \varepsilon) = \text{slim}(h, 0, \tau_1, \varepsilon) = \text{slim}(g, 0, \tau_1, \varepsilon)$.

4. Soft Limit at Infinity

This section defines the soft limit at infinity and investigates some basic properties. Throughout this section, let $\tau_\infty : \mathbb{R} \rightarrow P(\mathbb{R})$ be a function such that $\tau_\infty(K) \subseteq (K, \infty)$ and $\tau_\infty(K)$ is a set unbounded above, for all $K \in \mathbb{R}$. Similarly, let $\tau_{-\infty} : \mathbb{R} \rightarrow P(\mathbb{R})$ be a function such that $\tau_{-\infty}(K) \subseteq (-\infty, K)$ and $\tau_{-\infty}(K)$ is a set unbounded below, for all $K \in \mathbb{R}$. These descriptions also hold for $\lambda_\infty, \kappa_\infty, \lambda_{-\infty}$, and $\kappa_{-\infty}$, defined from $\mathbb{R}$ to $P(\mathbb{R})$. In addition, let $A \subseteq \mathbb{R}$ and $f, g : A \rightarrow \mathbb{R}$ be two functions unless stated otherwise.

Definition 15. Let A be a set unbounded above and $M \in \mathbb{R}$. If there exists a $K \in \mathbb{R}$ such that $\tau_\infty(K) \cap A \neq \emptyset$ and $x \in \tau_\infty(K) \cap A \Rightarrow |f(x) - M| \leq \varepsilon(K)$, then M is called a $(\tau_\infty(K), \varepsilon)$ -soft limit of f as $x \rightarrow \infty$. The set of all $(\tau_\infty(K), \varepsilon)$ -soft limits of f as $x \rightarrow \infty$ is denoted by $\text{slim}(f, \tau_\infty(K), \varepsilon)$.

Definition 16. Let A be a set unbounded below and $M \in \mathbb{R}$. If there exists a $K \in \mathbb{R}$ such that $\tau_{-\infty}(K) \cap A \neq \emptyset$ and $x \in \tau_{-\infty}(K) \cap A \Rightarrow |f(x) - M| \leq \varepsilon(K)$, then M is called a $(\tau_{-\infty}(K), \varepsilon)$ -soft limit of f as $x \rightarrow -\infty$. The set of all $(\tau_{-\infty}(K), \varepsilon)$ -soft limits of f as $x \rightarrow -\infty$ is denoted by $\text{slim}(f, \tau_{-\infty}(K), \varepsilon)$.

Definition 17. Let A be a set unbounded above and $K \in \mathbb{R}$. If

$$\sup_{x, y \in \tau_\infty(K) \cap A} \{f(x) - f(y)\} \in \mathbb{R}$$

then this real number is called soft $\tau_\infty(K)$ -variance of f and denoted by $\Delta(f, \tau_\infty(K))$.

Definition 18. Let A be a set unbounded below and $K \in \mathbb{R}$. If

$$\sup_{x, y \in \tau_{-\infty}(K) \cap A} \{f(x) - f(y)\} \in \mathbb{R}$$

then this real number is called soft $\tau_{-\infty}(K)$ -variance of f and denoted by $\Delta(f, \tau_{-\infty}(K))$.

Theorem 47. Let A be a set unbounded above, $K \in \mathbb{R}$, and $\tau_\infty(K) \cap A \neq \emptyset$. Then, $\text{slim}(f, \tau_\infty(K), \varepsilon) \neq \emptyset$ iff $\Delta(f, \tau_\infty(K)) \leq 2\varepsilon(K)$.

Theorem 48. Let A be a set unbounded below, $K \in \mathbb{R}$, and $\tau_{-\infty}(K) \cap A \neq \emptyset$. Then, $\text{slim}(f, \tau_{-\infty}(K), \varepsilon) \neq \emptyset$ iff $\Delta(f, \tau_{-\infty}(K)) \leq 2\varepsilon(K)$.

Theorem 49. Let A be a set unbounded above and $K \in \mathbb{R}$. If $\text{slim}(f, \tau_\infty(K), \varepsilon) \neq \emptyset$, then

$$\text{slim}(f, \tau_\infty(K), \varepsilon) = \left[\sup_{x \in \tau_\infty(K) \cap A} \{f(x)\} - \varepsilon(K), \inf_{x \in \tau_\infty(K) \cap A} \{f(x)\} + \varepsilon(K) \right]$$

Theorem 50. Let A be a set unbounded below and $K \in \mathbb{R}$. If $\text{slim}(f, \tau_{-\infty}(K), \varepsilon) \neq \emptyset$, then

$$\text{slim}(f, \tau_{-\infty}(K), \varepsilon) = \left[\sup_{x \in \tau_{-\infty}(K) \cap A} \{f(x)\} - \varepsilon(K), \inf_{x \in \tau_{-\infty}(K) \cap A} \{f(x)\} + \varepsilon(K) \right]$$

Example 15. For the function $f : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{2x+1}{x+1}$, $\tau_\infty(1) = (5, \infty)$, $\tau_{-\infty}(-3) = (-\infty, -3)$, $\varepsilon(1) = 2$, and $\varepsilon(-3) = \frac{3}{2}$,

$$\text{slim}(f, \tau_\infty(1), \varepsilon) = \left[\sup_{x \in \tau_\infty(1) \cap (\mathbb{R} \setminus \{-1\})} \left\{ \frac{2x+1}{x+1} \right\} - 2, \inf_{x \in \tau_\infty(1) \cap (\mathbb{R} \setminus \{-1\})} \left\{ \frac{2x+1}{x+1} \right\} + 2 \right] = \left[0, \frac{23}{6} \right]$$

Moreover,

$$\text{slim}(f, \tau_{-\infty}(-3), \varepsilon) = \left[\sup_{x \in \tau_{-\infty}(-3) \cap (\mathbb{R} \setminus \{-1\})} \left\{ \frac{2x+1}{x+1} \right\} - \frac{3}{2}, \inf_{x \in \tau_{-\infty}(-3) \cap (\mathbb{R} \setminus \{-1\})} \left\{ \frac{2x+1}{x+1} \right\} + \frac{3}{2} \right] = \left[1, \frac{7}{2} \right]$$

Theorem 51. Let A be a set unbounded above, $K \in \mathbb{R}$, and $\beta(K) \leq \alpha(K)$. If $\text{slim}(f, \tau_{\infty}(K), \beta) \neq \emptyset$, then $\text{slim}(f, \tau_{\infty}(K), \alpha) \neq \emptyset$. Moreover, $\text{slim}(f, \tau_{\infty}(K), \beta) \subseteq \text{slim}(f, \tau_{\infty}(K), \alpha)$.

Theorem 52. Let A be a set unbounded below, $K \in \mathbb{R}$, and $\beta(K) \leq \alpha(K)$. If $\text{slim}(f, \tau_{-\infty}(K), \beta) \neq \emptyset$, then $\text{slim}(f, \tau_{-\infty}(K), \alpha) \neq \emptyset$. Moreover, $\text{slim}(f, \tau_{-\infty}(K), \beta) \subseteq \text{slim}(f, \tau_{-\infty}(K), \alpha)$.

The proofs of Theorems 51 and 52 are similar to the proofs of Theorem 20.

Theorem 53. Let A be a set unbounded above, $K \in \mathbb{R}$, and $\emptyset \neq \lambda_{\infty}(K) \cap A \subseteq \tau_{\infty}(K) \cap A$. If $\text{slim}(f, \tau_{\infty}(K), \varepsilon) \neq \emptyset$, then $\text{slim}(f, \lambda_{\infty}(K), \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, \tau_{\infty}(K), \varepsilon) \subseteq \text{slim}(f, \lambda_{\infty}(K), \varepsilon)$.

Theorem 54. Let A be a set unbounded below, $K \in \mathbb{R}$, and $\emptyset \neq \lambda_{-\infty}(K) \cap A \subseteq \tau_{-\infty}(K) \cap A$. If $\text{slim}(f, \tau_{-\infty}(K), \varepsilon) \neq \emptyset$, then $\text{slim}(f, \lambda_{-\infty}(K), \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, \tau_{-\infty}(K), \varepsilon) \subseteq \text{slim}(f, \lambda_{-\infty}(K), \varepsilon)$.

The proofs of Theorems 53 and 54 are similar to the proofs of Theorem 25.

Theorem 55. Let A be a set unbounded above, $t, K, L, P \in \mathbb{R}$, and $t \neq 0$.

- i. $\text{slim}(f, \tau_{\infty}(K), \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) \neq \emptyset$. Moreover, $\text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) = t\text{slim}(f, \tau_{\infty}(K), \varepsilon)$.
- ii. If $\text{slim}(f, \tau_{\infty}(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_{\infty}(L), \beta) \neq \emptyset$, then $\text{slim}(f + g, \kappa_{\infty}(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{\infty}(P) \cap A \subseteq (\tau_{\infty}(K) \cap A) \cap (\lambda_{\infty}(L) \cap A)$ and $\alpha(K) + \beta(L) \leq \varepsilon(P)$. Moreover,

$$\text{slim}(f, \tau_{\infty}(K), \alpha) + \text{slim}(g, \lambda_{\infty}(L), \beta) \subseteq \text{slim}(f + g, \kappa_{\infty}(P), \varepsilon)$$

- iii. If $\text{slim}(f, \tau_{\infty}(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_{\infty}(L), \beta) \neq \emptyset$, then $\text{slim}(f - g, \kappa_{\infty}(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{\infty}(P) \cap A \subseteq (\tau_{\infty}(K) \cap A) \cap (\lambda_{\infty}(L) \cap A)$ and $\alpha(K) + \beta(L) \leq \varepsilon(P)$. Moreover,

$$\text{slim}(f, \tau_{\infty}(K), \alpha) - \text{slim}(g, \lambda_{\infty}(L), \beta) \subseteq \text{slim}(f - g, \kappa_{\infty}(P), \varepsilon)$$

Proof. Let A be a set unbounded above, $t, K, L, P \in \mathbb{R}$, and $t \neq 0$.

- i. $(\Rightarrow)$: Let $\text{slim}(f, \tau_{\infty}(K), \varepsilon) \neq \emptyset$. Then, there exists an $M \in \mathbb{R}$ such that

$$\tau_{\infty}(K) \cap A \neq \emptyset \text{ and } x \in \tau_{\infty}(K) \cap A \Rightarrow |f(x) - M| \leq \varepsilon(K)$$

Since

$$x \in \tau_{\infty}(K) \cap A \Rightarrow |(tf)(x) - tM| = |tf(x) - tM| = |t||f(x) - M| \leq |t|\varepsilon(K)$$

then $tM \in \text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon)$. That is, $\text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) \neq \emptyset$.

$(\Leftarrow)$: Let $\text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) \neq \emptyset$. Then, there exists an $M \in \mathbb{R}$ such that

$$\tau_{\infty}(K) \cap A \neq \emptyset \text{ and } x \in \tau_{\infty}(K) \cap A \Rightarrow |(tf)(x) - M| \leq |t|\varepsilon(K)$$

Since

$$x \in \tau_{\infty}(K) \cap A \Rightarrow \left| f(x) - \frac{M}{t} \right| = \frac{1}{|t|} |(tf)(x) - M| \leq \frac{1}{|t|} |t|\varepsilon(K) = \varepsilon(K)$$

then $\frac{M}{t} \in \text{slim}(f, \tau_{\infty}(K), \varepsilon)$. That is, $\text{slim}(f, \tau_{\infty}(K), \varepsilon) \neq \emptyset$.

Moreover, for $t > 0$,

$$\begin{aligned} \text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) &= \left[\sup_{x \in \tau_{\infty}(K) \cap A} \{(tf)(x)\} - |t|\varepsilon(K), \inf_{x \in \tau_{\infty}(K) \cap A} \{(tf)(x)\} + |t|\varepsilon(K) \right] \\ &= t \left[\sup_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} - \varepsilon(K), \inf_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} + \varepsilon(K) \right] \\ &= t\text{slim}(f, \tau_{\infty}(K), \varepsilon) \end{aligned}$$

and for $t < 0$,

$$\begin{aligned} \text{slim}(tf, \tau_{\infty}(K), |t|\varepsilon) &= \left[\sup_{x \in \tau_{\infty}(K) \cap A} \{(tf)(x)\} - |t|\varepsilon(K), \inf_{x \in \tau_{\infty}(K) \cap A} \{(tf)(x)\} + |t|\varepsilon(K) \right] \\ &= \left[t \inf_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} + t\varepsilon(K), t \sup_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} - t\varepsilon(K) \right] \\ &= t \left[\sup_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} - \varepsilon(K), \inf_{x \in \tau_{\infty}(K) \cap A} \{f(x)\} + \varepsilon(K) \right] \\ &= t\text{slim}(f, \tau_{\infty}(K), \varepsilon) \end{aligned}$$

ii. Let $\text{slim}(f, \tau_{\infty}(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_{\infty}(L), \beta) \neq \emptyset$. Then, there exist $M, N \in \mathbb{R}$ such that

$$\tau_{\infty}(K) \cap A \neq \emptyset \text{ and } x \in \tau_{\infty}(K) \cap A \Rightarrow |f(x) - M| \leq \alpha(K)$$

and

$$\lambda_{\infty}(L) \cap A \neq \emptyset \text{ and } x \in \lambda_{\infty}(L) \cap A \Rightarrow |g(x) - N| \leq \beta(L)$$

Since

$$\emptyset \neq \kappa_{\infty}(P) \cap A \subseteq (\tau_{\infty}(K) \cap A) \cap (\lambda_{\infty}(L) \cap A) \quad \text{and} \quad \alpha(K) + \beta(L) \leq \varepsilon(P)$$

then

$$x \in \kappa_{\infty}(P) \cap A \Rightarrow |(f+g)(x) - (M+N)| \leq |f(x) - M| + |g(x) - N| \leq \alpha(K) + \beta(L) \leq \varepsilon(P)$$

Thus, $M + N \in \text{slim}(f + g, \kappa_\infty(P), \varepsilon)$. That is, $\text{slim}(f + g, \kappa_\infty(P), \varepsilon) \neq \emptyset$. Moreover, for $S = \text{slim}(f, \tau_\infty(K), \alpha) + \text{slim}(g, \lambda_\infty(L), \beta)$,

$$\begin{aligned} S &= \left[\sup_{x \in \tau_\infty(K) \cap A} \{f(x)\} - \alpha(K), \inf_{x \in \tau_\infty(K) \cap A} \{f(x)\} + \alpha(K) \right] \\ &\quad + \left[\sup_{x \in \lambda_\infty(L) \cap A} \{g(x)\} - \beta(L), \inf_{x \in \lambda_\infty(L) \cap A} \{g(x)\} + \beta(L) \right] \\ &= \left[\sup_{x \in \tau_\infty(K) \cap A} \{f(x)\} - \alpha(K) + \sup_{x \in \lambda_\infty(L) \cap A} \{g(x)\} - \beta(L), \right. \\ &\quad \left. \inf_{x \in \tau_\infty(K) \cap A} \{f(x)\} + \alpha(K) + \inf_{x \in \lambda_\infty(L) \cap A} \{g(x)\} + \beta(L) \right] \\ &\subseteq \left[\sup_{x \in \kappa_\infty(P) \cap A} \{(f + g)(x)\} - \varepsilon(P), \inf_{x \in \kappa_\infty(P) \cap A} \{(f + g)(x)\} + \varepsilon(P) \right] \\ &= \text{slim}(f + g, \kappa_\infty(P), \varepsilon) \end{aligned}$$

iii. The proof can be observed from *i* and *ii*.

□

Theorem 56. Let A be a set unbounded below, $t, K, L, P \in \mathbb{R}$, and $t \neq 0$.

i. $\text{slim}(f, \tau_\infty(K), \varepsilon) \neq \emptyset$ iff $\text{slim}(tf, \tau_\infty(K), |t|\varepsilon) \neq \emptyset$. Moreover,

$$\text{slim}(tf, \tau_\infty(K), |t|\varepsilon) = t\text{slim}(f, \tau_\infty(K), \varepsilon)$$

ii. If $\text{slim}(f, \tau_\infty(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_\infty(L), \beta) \neq \emptyset$, then $\text{slim}(f + g, \kappa_\infty(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_\infty(P) \cap A \subseteq (\tau_\infty(K) \cap A) \cap (\lambda_\infty(L) \cap A)$ and $\alpha(K) + \beta(L) \leq \varepsilon(P)$. Moreover,

$$\text{slim}(f, \tau_\infty(K), \alpha) + \text{slim}(g, \lambda_\infty(L), \beta) \subseteq \text{slim}(f + g, \kappa_\infty(P), \varepsilon)$$

iii. If $\text{slim}(f, \tau_\infty(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_\infty(L), \beta) \neq \emptyset$, then $\text{slim}(f - g, \kappa_\infty(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_\infty(P) \cap A \subseteq (\tau_\infty(K) \cap A) \cap (\lambda_\infty(L) \cap A)$ and $\alpha(K) + \beta(L) \leq \varepsilon(P)$. Moreover,

$$\text{slim}(f, \tau_\infty(K), \alpha) - \text{slim}(g, \lambda_\infty(L), \beta) \subseteq \text{slim}(f - g, \kappa_\infty(P), \varepsilon)$$

The proof of Theorem 56 is similar to the proof of Theorem 55.

Theorem 57. Let A be a set unbounded above, $f(A), g(A) \subseteq \mathbb{R}^{\geq 0}$, and $K, L, P \in \mathbb{R}$. If $\text{slim}(f, \tau_\infty(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_\infty(L), \beta) \neq \emptyset$, then $\text{slim}(fg, \kappa_\infty(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_\infty(P) \cap A \subseteq (\tau_\infty(K) \cap A) \cap (\lambda_\infty(L) \cap A)$ and

$$\left(\sup_{x \in \tau_\infty(K) \cap A} \{f(x)\} + \alpha(K) \right) \left(\sup_{x \in \lambda_\infty(L) \cap A} \{g(x)\} + \beta(L) \right) \leq \varepsilon(P)$$

Moreover,

$$\text{slim}(f, \tau_\infty(K), \alpha) \text{slim}(g, \lambda_\infty(L), \beta) \subseteq \text{slim}(fg, \kappa_\infty(P), \varepsilon)$$

Theorem 58. Let A be a set unbounded above, $f(A), g(A) \subseteq \mathbb{R}^-$, and $K, L, P \in \mathbb{R}$. If $\text{slim}(f, \tau_\infty(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_\infty(L), \beta) \neq \emptyset$, then $\text{slim}(fg, \kappa_\infty(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_\infty(P) \cap A \subseteq (\tau_\infty(K) \cap A) \cap (\lambda_\infty(L) \cap A)$ and

$$\left(\inf_{x \in \tau_\infty(K) \cap A} \{f(x)\} - \alpha(K) \right) \left(\inf_{x \in \lambda_\infty(L) \cap A} \{g(x)\} - \beta(L) \right) \leq \varepsilon(P)$$

Moreover,

$$\text{slim}(f, \tau_\infty(K), \alpha) \text{slim}(g, \lambda_\infty(L), \beta) \subseteq \text{slim}(fg, \kappa_\infty(P), \varepsilon)$$

Theorem 59. Let A be a set unbounded below, $f(A), g(A) \subseteq \mathbb{R}^{\geq 0}$, and $K, L, P \in \mathbb{R}$. If $\text{slim}(f, \tau_{-\infty}(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_{-\infty}(L), \beta) \neq \emptyset$, then $\text{slim}(fg, \kappa_{-\infty}(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{-\infty}(P) \cap A \subseteq (\tau_{-\infty}(K) \cap A) \cap (\lambda_{-\infty}(L) \cap A)$ and

$$\left(\sup_{x \in \tau_{-\infty}(K) \cap A} \{f(x)\} + \alpha(K) \right) \left(\sup_{x \in \lambda_{-\infty}(L) \cap A} \{g(x)\} + \beta(L) \right) \leq \varepsilon(P)$$

Moreover,

$$\text{slim}(f, \tau_{-\infty}(K), \alpha) \text{slim}(g, \lambda_{-\infty}(L), \beta) \subseteq \text{slim}(fg, \kappa_{-\infty}(P), \varepsilon)$$

Theorem 60. Let A be a set unbounded below, $f(A), g(A) \subseteq \mathbb{R}^-$, and $K, L, P \in \mathbb{R}$. If $\text{slim}(f, \tau_{-\infty}(K), \alpha) \neq \emptyset$ and $\text{slim}(g, \lambda_{-\infty}(L), \beta) \neq \emptyset$, then $\text{slim}(fg, \kappa_{-\infty}(P), \varepsilon) \neq \emptyset$ such that $\emptyset \neq \kappa_{-\infty}(P) \cap A \subseteq (\tau_{-\infty}(K) \cap A) \cap (\lambda_{-\infty}(L) \cap A)$ and

$$\left(\inf_{x \in \tau_{-\infty}(K) \cap A} \{f(x)\} - \alpha(K) \right) \left(\inf_{x \in \lambda_{-\infty}(L) \cap A} \{g(x)\} - \beta(L) \right) \leq \varepsilon(P)$$

Moreover,

$$\text{slim}(f, \tau_{-\infty}(K), \alpha) \text{slim}(g, \lambda_{-\infty}(L), \beta) \subseteq \text{slim}(fg, \kappa_{-\infty}(P), \varepsilon)$$

The proofs of Theorems 57–60 are similar to the proofs of Theorems 40 and 41.

5. Comparisons of Classical and Soft Limit Types

This section compares the soft limit with classical limit. Moreover, it compares left / right / upper / lower soft limits with classical left-sided / right-sided / upper / lower limits. Throughout this section, let $A \subseteq \mathbb{R}$, $f : A \rightarrow \mathbb{R}$ be a function, and A' be the set of all accumulation points of A unless stated otherwise.

Theorem 61 ([4]). Let $a \in A'$. If there exists the classical limit of f at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau, \varepsilon) \subseteq \lim_{x \rightarrow a} f(x) + [-\varepsilon(a), \varepsilon(a)]$.

Example 16. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x - 1$, $\lim_{x \rightarrow 1} f(x) = 1$. Moreover,

$$\begin{aligned} \text{slim}(f, 1, \tau_\delta, \varepsilon) &= \left[\sup_{x \in \tau_\delta(1) \cap \mathbb{R}} \{2x - 1\} - \varepsilon(1), \inf_{x \in \tau_\delta(1) \cap \mathbb{R}} \{2x - 1\} + \varepsilon(1) \right] \\ &= [1 + 2\delta(1) - \varepsilon(1), 1 - 2\delta(1) + \varepsilon(1)] \neq \emptyset, \quad \delta(1) \leq \frac{\varepsilon(1)}{2} \end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} f(x) = 1 \in \text{slim}(f, 1, \tau_\delta, \varepsilon)$. Additionally, for $\delta(1) \leq \frac{\varepsilon(1)}{2}$,

$$\begin{aligned}\text{slim}(f, 1, \tau_\delta, \varepsilon) &= [1 + 2\delta(1) - \varepsilon(1), 1 - 2\delta(1) + \varepsilon(1)] \\ &\subseteq [1 - \varepsilon(1), 1 + \varepsilon(1)] \\ &= \lim_{x \rightarrow 1} f(x) + [-\varepsilon(1), \varepsilon(1)]\end{aligned}$$

To exemplify, for $\delta(1) = 1$ and $\varepsilon(1) = 4$,

$$\text{slim}(f, 1, \tau_1, 4) = [-1, 3] \subseteq [-3, 5] = \lim_{x \rightarrow 1} f(x) + [-\varepsilon(1), \varepsilon(1)]$$

The converse of Theorem 61 is not always correct (see Examples 17 and 18).

Example 17. For the function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{|x|}{x}$ and $\varepsilon(0) \geq 1$, $\text{slim}(f, 0, \tau_\delta, \varepsilon) \neq \emptyset$. Here, $\text{slim}(f, 0, \tau_\delta, \varepsilon) = [1 - \varepsilon(0), -1 + \varepsilon(0)]$. On the other hand, there is no classical limit of f at $a = 0$.

Example 18. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \text{sgn}(x)$, the signum function, and $\varepsilon(0) \geq 1$, $\text{slim}(f, 0, \tau_\delta, \varepsilon) \neq \emptyset$. Here, $\text{slim}(f, 0, \tau_\delta, \varepsilon) = [1 - \varepsilon(0), -1 + \varepsilon(0)]$. On the other hand, there is no classical limit of f at $a = 0$.

Theorem 62. Let $(a - d, a) \cap A \neq \emptyset$, for all $d > 0$. If there exists the classical left-sided limit of f at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $\text{slim}(f, a, \tau^-, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^-, \varepsilon) \subseteq \lim_{x \rightarrow a^-} f(x) + [-\varepsilon(a), \varepsilon(a)]$.

Theorem 63. Let $(a, a + d) \cap A \neq \emptyset$, for all $d > 0$. If there exists the classical right-sided limit of f at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $\text{slim}(f, a, \tau^+, \varepsilon) \neq \emptyset$. Moreover, $\text{slim}(f, a, \tau^+, \varepsilon) \subseteq \lim_{x \rightarrow a^+} f(x) + [-\varepsilon(a), \varepsilon(a)]$.

Theorem 64. Let $a \in A'$. If there exists the classical upper limit of f at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$.

Theorem 65. Let $a \in A'$. If there exists the classical lower limit of f at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$.

6. Soft Continuity

This section presents soft and upper (lower) soft continuity provided in [3–5] and theoretically contributes to these concepts. Throughout this section, let $A, B \subseteq \mathbb{R}$, $a \in A$, $t \in \mathbb{R}$, and $f, g : A \rightarrow \mathbb{R}$ be two functions unless stated otherwise.

Definition 19 ([3,4]). Let $\text{slim}(f, a, \tau, \varepsilon) \neq \emptyset$. If $\tau(a) \subseteq A$ and $f(a) \in \text{slim}(f, a, \tau, \varepsilon)$, then f is said to be (τ, ε) -soft continuous at a . Moreover, if f is (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be (τ, ε) -soft continuous on X . In addition, if $\tau(a) = \tau_\delta(a)$, then f is said to be $(\tau_\delta, \varepsilon)$ -soft continuous at a .

Definition 20 ([4]). Let $\overline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$. If $\tau(a) \subseteq A$ and $f(a) \in \overline{\text{slim}}(f, a, \tau, \varepsilon)$, then f is said to be upper (τ, ε) -soft continuous at a .

Definition 21 ([4]). Let $\underline{\text{slim}}(f, a, \tau, \varepsilon) \neq \emptyset$. If $\tau(a) \subseteq A$ and $f(a) \in \underline{\text{slim}}(f, a, \tau, \varepsilon)$, then f is said to be lower (τ, ε) -soft continuous at a .

Note 7. This study uses Definition 22 instead of Definition 19 to make the relationship between soft continuity and soft limit more useful. Similarly, it uses Definitions 23 and 24 instead of Definitions 20 and 21, respectively. Furthermore, this study rearranges the related properties provided in [3–5] according to Definitions 22–24.

Definition 22. If $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$, then f is said to be (τ, ε) -soft continuous at a . Moreover, if f is (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be (τ, ε) -soft continuous on X .

Definition 23. If $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(x) \leq f(a) + \varepsilon(a)$, then f is said to be upper (τ, ε) -soft continuous at a . Moreover, if f is upper (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be upper (τ, ε) -soft continuous on X .

Definition 24. If $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(a) - \varepsilon(a) \leq f(x)$, then f is said to be lower (τ, ε) -soft continuous at a . Moreover, if f is lower (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be lower (τ, ε) -soft continuous on X .

Example 19. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$ and for $\tilde{\tau}_3(2) \cap \mathbb{R} = [-1, 5] \cap \mathbb{R} = [-1, 5]$, since

$$\begin{aligned} \forall x \in \tilde{\tau}_3(2) \cap \mathbb{R}, -1 \leq x \leq 5 &\Rightarrow \forall x \in \tilde{\tau}_3(2) \cap \mathbb{R}, -3 \leq x + 3 - 5 \leq 3 \\ &\Rightarrow \forall x \in \tilde{\tau}_3(2) \cap \mathbb{R}, |f(x) - f(2)| \leq 3 \leq 4 = \varepsilon(2) \end{aligned}$$

then f is (τ_3, ε) -soft continuous at $a = 2$.

In addition, left (τ, ε) -soft continuity and right (τ, ε) -soft continuity are defined as follows:

Definition 25. If $x \in \tilde{\tau}_{D(f)}^-(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$, then f is said to be left (τ, ε) -soft continuous at a . Moreover, if f is left (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be left (τ, ε) -soft continuous on X .

Definition 26. If $x \in \tilde{\tau}_{D(f)}^+(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$, then f is said to be right (τ, ε) -soft continuous at a . Moreover, if f is right (τ, ε) -soft continuous at all points of $X \subseteq A$, then f is said to be right (τ, ε) -soft continuous on X .

Theorem 66. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $f(a) \in \text{slim}(f, a, \tau, \varepsilon)$ iff f is (τ, ε) -soft continuous at a .

Proof. Let $\tau_{D(f)}(a) \neq \emptyset$.

($\Rightarrow$): Let $f(a) \in \text{slim}(f, a, \tau, \varepsilon)$. Then, $x \in \tau_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Moreover, if $x = a$, then $|f(a) - f(a)| \leq \varepsilon(a)$. Therefore, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Thus, f is (τ, ε) -soft continuous at a .

($\Leftarrow$): Let f be (τ, ε) -soft continuous at a . Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. In addition,

$$\begin{aligned} x \in \tau_{D(f)}(a) &\Rightarrow x \in \tilde{\tau}_{D(f)}(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \end{aligned}$$

Thus, $f(a) \in \text{slim}(f, a, \tau, \varepsilon)$. $\square$

Theorem 67. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $f(a) \in \overline{\text{slim}}(f, a, \tau, \varepsilon)$ iff f is upper (τ, ε) -soft continuous at a .

Theorem 68. Let $\tau_{D(f)}(a) \neq \emptyset$. Then, $f(a) \in \underline{\text{slim}}(f, a, \tau, \varepsilon)$ iff f is lower (τ, ε) -soft continuous at a .

Theorem 69. Let $\tau_{D(f)}^-(a) \neq \emptyset$. Then, $f(a) \in \text{slim}(f, a, \tau^-, \varepsilon)$ iff f is left (τ, ε) -soft continuous at a .

Theorem 70. Let $\tau_{D(f)}^+(a) \neq \emptyset$. Then, $f(a) \in \text{slim}(f, a, \tau^+, \varepsilon)$ iff f is right (τ, ε) -soft continuous at a .

The proofs of Theorems 67–70 are as in Theorem 66.

Theorem 71. f is left (τ, ε) -soft continuous and right (τ, ε) -soft continuous at a iff f is (τ, ε) -soft continuous at a .

Proof. $(\Rightarrow)$: Let f be left (τ, ε) -soft continuous and right (τ, ε) -soft continuous at a . Then,

$$x \in \tilde{\tau}_{D(f)}^-(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \quad \text{and} \quad x \in \tilde{\tau}_{D(f)}^+(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$$

Therefore, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Consequently, f is (τ, ε) -soft continuous at a .

$(\Leftarrow)$: Let f be (τ, ε) -soft continuous at a . Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Therefore,

$$\begin{aligned} x \in \tilde{\tau}_{D(f)}^-(a) &\Rightarrow x \in \tilde{\tau}_{D(f)}(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \end{aligned}$$

and

$$\begin{aligned} x \in \tilde{\tau}_{D(f)}^+(a) &\Rightarrow x \in \tilde{\tau}_{D(f)}(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \end{aligned}$$

Consequently, f is left (τ, ε) -soft continuous and right (τ, ε) -soft continuous at a . $\square$

Theorem 72. f is upper (τ, ε) -soft continuous and lower (τ, ε) -soft continuous at a iff f is (τ, ε) -soft continuous at a .

The proof is as in Theorem 71.

Since some of the conditions in Proposition D.1.1 ([4], p. 231) are rearranged as in Theorems 73 and 74, then this study proves one of them.

Theorem 73 ([4], p. 231). Let $X \subseteq A$. If f is upper (τ, ε) -soft continuous on X , then f is lower (τ^*, ε^*) -soft continuous on $\tilde{\tau}_{D(f)}(X) = \bigcup_{x \in X} \tilde{\tau}_{D(f)}(x) \subseteq A$ such that $\tau^*(a) = \{x \in X : a \in \tau(x)\}$, $\tilde{\tau}^*(a) = \tau^*(a) \cup \{a\}$, and $\varepsilon^*(a) = \sup_{x \in \tilde{\tau}^*(a)} \{\varepsilon(x)\} \in \mathbb{R}^{\geq 0}$, for all $a \in \tilde{\tau}_{D(f)}(X)$.

Proof. Let $X \subseteq A$ and f be upper (τ, ε) -soft continuous on X . Therefore, for all $x \in X$,

$$a \in \tilde{\tau}_{D(f)}(x) \Rightarrow f(a) \leq f(x) + \varepsilon(x)$$

For all $x \in X$, since $x \in \tilde{\tau}(x)$ and $x \in A$, then $\tilde{\tau}_{D(f)}(X) \neq \emptyset$. Let $a \in \tilde{\tau}_{D(f)}(X) = \bigcup_{x \in X} \tilde{\tau}_{D(f)}(x) \subseteq A$. Moreover, since $\tilde{\tau}^*(a) = \tau^*(a) \cup \{a\}$ and $a \in A$, then $\tilde{\tau}_{D(f)}^*(a) \neq \emptyset$. Hence,

$$\begin{aligned}
 x \in \tilde{\tau}_{D(f)}^*(a) &\Rightarrow x \in \tilde{\tau}^*(a) \\
 &\Rightarrow x \in \tau^*(a) \cup \{a\} \\
 &\Rightarrow x \in \tau^*(a) \vee x = a \\
 &\Rightarrow (x \in X \wedge a \in \tau(x)) \vee a = x \\
 &\Rightarrow (x \in X \wedge a \in \tilde{\tau}_f(x)) \vee a = x, & \tau(x) \subseteq \tilde{\tau}(x) \wedge a \in A \\
 &\Rightarrow f(a) \leq f(x) + \varepsilon(x) \vee f(a) = f(x), & f \text{ is an upper } (\tau, \varepsilon)\text{-soft continuous function on } X \\
 &\Rightarrow f(a) \leq f(x) + \varepsilon(x) \vee f(a) \leq f(x) + \varepsilon(x), \varepsilon(x) \geq 0 \\
 &\Rightarrow f(a) \leq f(x) + \varepsilon(x) \\
 &\Rightarrow f(a) - \varepsilon^*(a) \leq f(a) - \varepsilon(x) \leq f(x), & \varepsilon^*(a) = \sup_{x \in \tilde{\tau}^*(a)} \{\varepsilon(x)\} \in \mathbb{R}^{\geq 0}, \text{ for all } a \in \tilde{\tau}_{D(f)}(X) \\
 &\Rightarrow f(a) - \varepsilon^*(a) \leq f(x)
 \end{aligned}$$

Therefore, $x \in \tilde{\tau}_{D(f)}^*(a) \Rightarrow f(a) - \varepsilon^*(a) \leq f(x)$, for all $a \in \tilde{\tau}_{D(f)}(X)$. Consequently, f is lower (τ^*, ε^*) -soft continuous on $\tilde{\tau}_{D(f)}(X)$. $\square$

Theorem 74 ([4], p. 231). Let $X \subseteq A$. If f is lower (τ, ε) -soft continuous on X , then f is upper (τ^*, ε^*) -soft continuous on $\tilde{\tau}_{D(f)}(X) = \bigcup_{x \in X} \tilde{\tau}_{D(f)}(x) \subseteq A$ such that $\tau^*(a) = \{x \in X : a \in \tau(x)\}$, $\tilde{\tau}^*(a) = \tau^*(a) \cup \{a\}$, and $\varepsilon^*(a) = \sup_{x \in \tilde{\tau}^*(a)} \{\varepsilon(x)\}$, for all $a \in \tilde{\tau}_{D(f)}(X)$.

The proof is as in Theorem 73.

Example 20. For the function $f : A \rightarrow \mathbb{R}$ defined by $f(x) = c$ such that $\tau_{D(f)}(a) \neq \emptyset$ and $a \in A$, $\text{slim}(f, a, \tau, \varepsilon) = [c - \varepsilon(a), c + \varepsilon(a)]$ and $f(a) = c \in \text{slim}(f, a, \tau, \varepsilon)$. Therefore, from Theorem 66, f is (τ, ε) -soft continuous at any point $a \in A$ satisfying the condition $\tau_{D(f)}(a) \neq \emptyset$.

Example 21. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$, $\text{slim}(f, 2, \tau_\delta^-, \varepsilon) = [1, 6]$ and $\text{slim}(f, 2, \tau_\delta^+, \varepsilon) = [4, 9]$ such that $\delta(2) = 3$ and $\varepsilon(2) = 4$. Since $f(2) = 5 \in \text{slim}(f, 2, \tau_\delta^-, \varepsilon)$ and $f(2) = 5 \in \text{slim}(f, 2, \tau_\delta^+, \varepsilon)$, then f is left $(\tau_\delta, \varepsilon)$ - and right $(\tau_\delta, \varepsilon)$ -soft continuous at $a = 2$. It can also be observed that f is $(\tau_\delta, \varepsilon)$ -soft continuous at $a = 2$. Moreover, $\overline{\text{slim}}(f, 2, \tau_\delta, \varepsilon) = [4, \infty)$ and $\underline{\text{slim}}(f, 2, \tau_\delta, \varepsilon) = (-\infty, 6]$. As $f(2) = 5 \in \overline{\text{slim}}(f, 2, \tau_\delta, \varepsilon)$ and $f(2) = 5 \in \underline{\text{slim}}(f, 2, \tau_\delta, \varepsilon)$, then f is upper $(\tau_\delta, \varepsilon)$ - and lower $(\tau_\delta, \varepsilon)$ -soft continuous at $a = 2$.

Theorem 75. There exists a function $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ such that f is (τ, ε) -soft continuous at a .

Proof. For a function $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ such that $\tau_{D(f)}(a) = \emptyset$,

$$\begin{aligned}
 x \in \tilde{\tau}_{D(f)}(a) &\Rightarrow x \in \{a\} \\
 &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a)
 \end{aligned}$$

Then, f is (τ, ε) -soft continuous at a . $\square$

Note 8. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$, the floor function, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x , is not classical continuous for integers. On the other hand, from Theorem 75, f is (τ, ε) -soft continuous on $\mathbb{R}$, for the function τ defined by $\tau(x) = \emptyset$, for all $x \in \mathbb{R}$. Thus, every function is (τ, ε) -soft continuous for at least one function τ but may not be classical continuous.

Example 22. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \begin{cases} x, & x \geq 1 \\ x-1, & x < 1 \end{cases}$, $X = [1, 3]$, and $\delta(x) = \varepsilon(x) = 1$, for all $x \in \mathbb{R}$. Therefore, for all $a \in X$,

$$\begin{aligned} x \in \tilde{\tau}_\delta(a) \cap \mathbb{R} &\Rightarrow x \in [a-1, a+1] \\ &\Rightarrow f(x) \leq \sup_{x \in [a-1, a+1]} f(x) = a+1 = f(a) + \varepsilon(a) \end{aligned}$$

Hence, f is upper $(\tau_\delta, \varepsilon)$ -soft continuous on X . Then, from Theorem 73, f is lower (τ^*, ε^*) -soft continuous on $\bigcup_{x \in X} (\tilde{\tau}_\delta(x) \cap \mathbb{R}) = [0, 4]$ such that $\varepsilon^*(a) = 1$ and $\tau^*(a) = ([a-1, a] \cup (a, a+1]) \cap [1, 3]$, for all $a \in \bigcup_{x \in X} (\tilde{\tau}_\delta(x) \cap \mathbb{R})$. Thus, since $\tilde{\tau}^*(a) = [a-1, a+1] \cap [1, 3]$, then, for all $a \in \bigcup_{x \in X} (\tilde{\tau}_\delta(x) \cap \mathbb{R})$,

$$\begin{aligned} x \in \tilde{\tau}_{D(f)}^*(a) &\Rightarrow x \in [a-1, a+1] \cap [1, 3] \\ &\Rightarrow \begin{cases} f(a) - \varepsilon^*(a) \leq a-1 \leq f(1) \leq f(x), & a \in [0, 2) \\ f(a) - \varepsilon^*(a) = a-1 \leq f(a-1) \leq f(x), & a \in [2, 4] \end{cases} \end{aligned}$$

Note 9. It can be observed that the function f in Example 22 is not lower semicontinuous at $a = 1$. On the other hand, it is lower (τ^*, ε^*) -soft continuous at $a = 1$. That is, Example 22 shows the existence of a function, lower soft continuous but not lower semicontinuous. Similar explanations can also be observed for the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} x, & x > 1 \\ x-1, & x \leq 1 \end{cases}$.

Theorem 76. If f is (τ, ε) -soft continuous at a , then f is bounded on $\tilde{\tau}_{D(f)}(a)$.

Proof. Let f be (τ, ε) -soft continuous at a . Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(a) - \varepsilon(a) \leq f(x) \leq f(a) + \varepsilon(a)$$

Thus, $|f(x)| \leq \max\{|f(a) - \varepsilon(a)|, |f(a) + \varepsilon(a)|\}$, for all $x \in \tilde{\tau}_{D(f)}(a)$. Therefore, f is bounded on $\tilde{\tau}_{D(f)}(a)$. $\square$

Theorem 77. Let $\beta(a) \leq \alpha(a)$. If f is (τ, β) -soft continuous at a , then f is (τ, α) -soft continuous at a .

Proof. Let $\beta(a) \leq \alpha(a)$ and f be (τ, β) -soft continuous at a . Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \beta(a) \leq \alpha(a)$. Therefore, f is (τ, α) -soft continuous at a . $\square$

Theorem 78. Let $\lambda(a) \subseteq \tau(a)$. If f is (τ, ε) -soft continuous at a , then f is (λ, ε) -soft continuous at a .

Proof. Let $\lambda(a) \subseteq \tau(a)$ and f be (τ, ε) -soft continuous at a . Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Thus,

$$\begin{aligned} x \in \tilde{\lambda}_{D(f)}(a) &\Rightarrow x \in \tilde{\tau}_{D(f)}(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \end{aligned}$$

Therefore, f is (λ, ε) -soft continuous at a . $\square$

Theorem 79. Let $t \neq 0$. Then, f is (τ, ε) -soft continuous at a iff tf is $(\tau, |t|\varepsilon)$ -soft continuous at a .

Proof. Let $t \neq 0$.

($\Rightarrow$): Let f be (τ, ε) -soft continuous at a . Since $\tilde{\tau}_{D(f)}(a) = \tilde{\tau}_{D(tf)}(a)$, then

$$\begin{aligned} x \in \tilde{\tau}_{D(tf)}(a) &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \\ &\Rightarrow |(tf)(x) - (tf)(a)| \leq |t|\varepsilon(a) \end{aligned}$$

Therefore, tf is $(\tau, |t|\varepsilon)$ -soft continuous at a .

($\Leftarrow$): Let tf be $(\tau, |t|\varepsilon)$ -soft continuous at a . Since $\tilde{\tau}_{D(f)}(a) = \tilde{\tau}_{D(tf)}(a)$, then

$$\begin{aligned} x \in \tilde{\tau}_{D(f)}(a) &\Rightarrow |(tf)(x) - (tf)(a)| \leq |t|\varepsilon(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon(a) \end{aligned}$$

Therefore, f is (τ, ε) -soft continuous at a . $\square$

Theorem 80. If f and g are (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively, then $f + g$ is (κ, ε) -soft continuous at a such that $\kappa(a) \subseteq \tau(a) \cap \lambda(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$.

Proof. Let f and g be (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively. Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \alpha(a) \quad \text{and} \quad x \in \tilde{\lambda}_{D(g)}(a) \Rightarrow |g(x) - g(a)| \leq \beta(a)$$

Thus,

$$x \in \tilde{\tau}_{D(f)}(a) \cap \tilde{\lambda}_{D(g)}(a) \Rightarrow |f(x) - f(a)| + |g(x) - g(a)| \leq \alpha(a) + \beta(a) \leq \varepsilon(a)$$

Since $\kappa(a) \subseteq \tau(a) \cap \lambda(a)$, then

$$x \in \tilde{\kappa}_{D(f+g)}(a) \Rightarrow |(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| \leq \varepsilon(a)$$

Therefore, $f + g$ is (κ, ε) -soft continuous at a . $\square$

Corollary 19. If f and g are (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively, then $f - g$ is (κ, ε) -soft continuous at a such that $\kappa(a) \subseteq \tau(a) \cap \lambda(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$.

The proof of Corollary 19 can be observed from Theorems 79 and 80. Moreover, Theorems 77–80 and Corollary 19 are also valid for upper, lower, left, and right soft continuity.

Example 23. For the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$ and $g(x) = x^2$, respectively, and for $\alpha(1) = 5$, $\beta(1) = 9$, and $\tau_\delta(1) \cap \mathbb{R}$ such that $\delta(1) = 2$, $\text{slim}(f, 1, \tau_\delta, \alpha) = [1, 7]$, $\text{slim}(g, 1, \tau_\delta, \beta) = [0, 9]$, $\text{slim}(f + g, 1, \tau_\delta, \alpha + \beta) = [1, \frac{67}{4}]$, and $\text{slim}(f - g, 1, \tau_\delta, \alpha + \beta) = [-\frac{43}{4}, 11]$. Moreover, $f(1) = 4 \in \text{slim}(f, 1, \tau_\delta, \alpha)$ and $g(1) = 1 \in \text{slim}(g, 1, \tau_\delta, \beta)$. Therefore, from Theorem 66, f and g are (τ_δ, α) -soft continuous and (τ_δ, β) -soft continuous at $a = 1$, respectively. In addition, $(f + g)(1) = 5 \in \text{slim}(f + g, 1, \tau_\delta, \alpha + \beta)$ and $(f - g)(1) = 3 \in \text{slim}(f - g, 1, \tau_\delta, \alpha + \beta)$. Thus, from Theorem 66, $f + g$ and $f - g$ are $(\tau_\delta, \alpha + \beta)$ -soft continuous at $a = 1$.

Theorem 81. If f and g are (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively, then fg is (κ, ε) -soft continuous at a such that $\kappa(a) \subseteq \tau(a) \cap \lambda(a)$ and $|g(a)|\alpha(a) + \max\{|f(a) - \alpha(a)|, |f(a) + \alpha(a)|\}\beta(a) \leq \varepsilon(a)$.

Proof. Let f and g be (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively. Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \alpha(a) \quad \text{and} \quad x \in \tilde{\lambda}_{D(g)}(a) \Rightarrow |g(x) - g(a)| \leq \beta(a)$$

Since $|(fg)(x) - (fg)(a)| \leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)|$, then, from Theorem 76,

$$\begin{aligned} x \in \tilde{\kappa}_{D(f)}(a) &\Rightarrow |(fg)(x) - (fg)(a)| \leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\ &\Rightarrow |(fg)(x) - (fg)(a)| \leq \max\{|f(a) - \alpha(a)|, |f(a) + \alpha(a)|\}\beta(a) + |g(a)|\alpha(a) \\ &\Rightarrow |(fg)(x) - (fg)(a)| \leq \varepsilon(a) \end{aligned}$$

Therefore, fg is (κ, ε) -soft continuous at a . $\square$

Theorem 82. Let $m \in \mathbb{R}^+$. If f is (τ, α) -soft continuous at a such that $|f(x)| > m$, for all $x \in \tilde{\tau}_{D(f)}(a)$, then $\frac{1}{f}$ is (τ, β) -soft continuous at a such that $\frac{\alpha(a)}{m|f(a)|} \leq \beta(a)$.

Proof. Let $m \in \mathbb{R}^+$ and f be (τ, α) -soft continuous at a such that $|f(x)| > m$, for all $x \in \tilde{\tau}_{D(f)}(a)$. Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \alpha(a)$. Since $\tilde{\tau}_{D(f)}(a) = \tilde{\tau}_{D(\frac{1}{f})}(a)$, then

$$x \in \tilde{\tau}_{D(\frac{1}{f})}(a) \Rightarrow \left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| \leq \frac{\alpha(a)}{m|f(a)|} \leq \beta(a)$$

Therefore, f is (τ, β) -soft continuous at a . $\square$

Corollary 20. Let $m \in \mathbb{R}^+$. If f and g are (τ, α) -soft continuous and (λ, β) -soft continuous at a , respectively, such that $|g(x)| > m$, for all $x \in \tilde{\lambda}_{D(g)}(a)$, then $\frac{f}{g}$ is (κ, ε) -soft continuous at a such that $\kappa(a) \subseteq \tau(a) \cap \lambda(a)$ and

$$\frac{1}{|g(a)|}\alpha(a) + \max\{|f(a) - \alpha(a)|, |f(a) + \alpha(a)|\}\frac{\beta(a)}{m|g(a)|} \leq \varepsilon(a)$$

Theorem 83. Let $\emptyset \neq B \subseteq A$ and $a \in B$. If f is (τ, ε) -soft continuous at a , then $f|_B$ is (τ, ε) -soft continuous at a .

Proof. Let $\emptyset \neq B \subseteq A$, $a \in B$, and f be (τ, ε) -soft continuous at a . Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$$

Thus,

$$x \in \tilde{\tau}_{D(f|_B)}(a) \Rightarrow |f|_B(x) - f|_B(a)| = |f(x) - f(a)| \leq \varepsilon(a)$$

Therefore, $f|_B$ is (τ, ε) -soft continuous at a . $\square$

Proposition 4. For the identity function $Id_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Id_{\mathbb{R}}(x) = x$ and for $\delta(a) \leq \varepsilon(a)$, $Id_{\mathbb{R}}$ is $(\tau_{\delta}, \varepsilon)$ -soft continuous at a .

Proof. For the identity function $Id_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Id_{\mathbb{R}}(x) = x$ and for $\delta(a) \leq \varepsilon(a)$, $Id_{\mathbb{R}}(a) = a \in \text{slim}(Id_{\mathbb{R}}, a, \tau_{\delta}, \varepsilon) = [a - (\varepsilon(a) - \delta(a)), a + (\varepsilon(a) - \delta(a))]$. Therefore, $Id_{\mathbb{R}}$ is $(\tau_{\delta}, \varepsilon)$ -soft continuous at a from Theorem 66. $\square$

Theorem 84. Let $g : B \rightarrow \mathbb{R}$ be a function, $f(A) \subseteq B$, and $\beta(f(a)) \leq \alpha(a)$. If f is (τ, α) -soft continuous at a , g is (λ, β) -soft continuous at $f(a)$, and $f(\tilde{\tau}_{D(f)}(a)) \subseteq \tilde{\lambda}_{D(g)}(f(a))$, then $g \circ f$ is (τ, α) -soft continuous at a .

Proof. Let $g : B \rightarrow \mathbb{R}$ be a function, $f(A) \subseteq B$, $\beta(f(a)) \leq \alpha(a)$, f be (τ, α) -soft continuous at a , g be (λ, β) -soft continuous at $f(a)$, and $f(\tilde{\tau}_{D(f)}(a)) \subseteq \tilde{\lambda}_{D(g)}(f(a))$. Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \alpha(a) \quad \text{and} \quad x \in \tilde{\lambda}_{D(g)}(f(a)) \Rightarrow |g(x) - g(f(a))| \leq \beta(f(a))$$

Thus,

$$\begin{aligned} x \in \tilde{\tau}_{D(g \circ f)}(a) &\Rightarrow f(x) \in f(\tilde{\tau}_{D(g \circ f)}(a)) \\ &\Rightarrow f(x) \in f(\tilde{\tau}_{D(f)}(a)) \\ &\Rightarrow f(x) \in \tilde{\lambda}_{D(g)}(f(a)) \\ &\Rightarrow |(g \circ f)(x) - (g \circ f)(a)| \leq \beta(f(a)) \leq \alpha(a) \end{aligned}$$

Therefore, $g \circ f$ is (τ, α) -soft continuous at a . $\square$

Theorem 85. Let $a \in A \cap B$, $g : B \rightarrow \mathbb{R}$ be a function, $\tau(a) \subseteq A \cap B$, and $f(x) = g(x)$, for all $x \in \tilde{\tau}(a)$. Then, f is (τ, ε) -soft continuous at a iff g is (τ, ε) -soft continuous at a .

Proof. Let $a \in A \cap B$, $g : B \rightarrow \mathbb{R}$ be a function, $\tau(a) \subseteq A \cap B$, and $f(x) = g(x)$, for all $x \in \tilde{\tau}(a)$.

($\Rightarrow$): Assume that f is (τ, ε) -soft continuous at a . Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$$

Since $\tau_{D(f)}(a) = \tau_{D(g)}(a) = \tau(a)$ and $a \in A \cap B$, then

$$x \in \tilde{\tau}_{D(g)}(a) \Rightarrow |g(x) - g(a)| = |f(x) - f(a)| \leq \varepsilon(a)$$

Hence, g is (τ, ε) -soft continuous at a .

($\Leftarrow$): The proof is similar. $\square$

Definition 27 ([3]). If $|f(a)| > \varepsilon(a)$, then $f(a)$ is called an ε -large value, and if $|f(a)| \leq \varepsilon(a)$, then $f(a)$ is called an ε -small value.

Example 24. Let $f : [-1, 3] \rightarrow \mathbb{R}$ be a function defined by $f(x) = 2x - 3$, $\varepsilon(-1) = 2$, and $\varepsilon(2) = 3$. Since $|f(-1)| = |-5| > \varepsilon(-1)$, then $f(-1)$ is an ε -large value. Furthermore, since $|f(2)| = 1 \leq \varepsilon(2)$, then $f(2)$ is an ε -small value.

Theorem 86 (Bolzano's Theorem for soft continuous functions [3]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a $(\tau_\delta, \varepsilon)$ -soft continuous function on $[a, b]$. If f has ε -large values on the ends of $[a, b]$ and the signs of these values are different, then there exists a point $c \in [a, b]$ such that $f(c)$ is an ε -small value.

The concept of (τ, ε) -continuity is analog to the classical continuity, and the concepts of left and right (τ, ε) -soft continuity are analogs of classical one-sided continuity. For classical one-sided continuous functions, Bolzano's Theorem is not valid. However, an analog of Theorem 86 holds for left and right (τ, ε) -soft continuous functions under certain conditions.

Theorem 87 (Bolzano's Theorem for one-sided soft continuous functions [3]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a right $(\tau_\delta, \varepsilon)$ -soft continuous (left $(\tau_\delta, \varepsilon)$ -soft continuous) function on $[a, b]$, f has ε -

large values on the ends of $[a, b]$, and these values have different signs. Moreover, let $\inf_{x \in [a, b]} \delta(x) > 0$. Then, there exists a point $c \in [a, b]$ such that $f(c)$ is an ε -small value.

Definition 28 ([3]). Let $L \in \mathbb{R}$. If $|f(a) - L| \leq \varepsilon(a)$, i.e., the difference between $f(a)$ and L is ε -small, then it is said to be $f(a)$ is ε -equal to L and denoted by $f(a) =_{\varepsilon} L$.

Example 25. For the function f and $\varepsilon(2) = 3$ in Example 24, since $|f(2) - (-1)| = |2| \leq 3$, then $f(2) = 1$ is ε -equal to -1 . It can also be observed that $f(2)$ is ε -equal to a , for all $a \in [-2, 4]$.

Theorem 88 (Intermediate Value Theorem for soft continuous functions [3]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a $(\tau_{\delta}, \varepsilon)$ -soft continuous function on $[a, b]$ such that $f(a) \neq f(b)$. If C is any real number between $f(a)$ and $f(b)$, then there exists a point $c \in [a, b]$ such that $f(c) =_{\varepsilon} C$.

Theorem 89 ([3]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a $(\tau_{\delta}, \varepsilon)$ -soft continuous function on $[a, b]$. Then, f is bounded on $[a, b]$.

Theorem 90 ([3]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a right $(\tau_{\delta}, \varepsilon)$ -soft continuous (left $(\tau_{\delta}, \varepsilon)$ -soft continuous) function on $[a, b]$ and $\inf_{x \in [a, b]} \delta(x) > 0$. Then, f is bounded on $[a, b]$.

Theorem 91 ([3]). Let f be bounded on A and $\inf_{a \in A} \delta(a) > 0$. Then, there exist $x, y \in A$ such that $f(x) =_{\delta} \sup_A f$ and $f(y) =_{\delta} \inf_A f$.

Example 26. For the function $f : [2, 6] \rightarrow \mathbb{R}$ defined by $f(x) = 3x - 5$ and $\delta(x) = |x + 1|$, f is bounded on $[2, 6]$ and $\inf_{x \in [2, 6]} \delta(x) = 3 > 0$. Moreover, $\sup_{[2, 6]} f = 13$ and $\inf_{[2, 6]} f = 1$. Thus, for $3, \frac{11}{2} \in [2, 6]$,

$$\left| f\left(\frac{11}{2}\right) - \sup_{[2, 6]} f \right| = \left| \frac{23}{2} - 13 \right| = \frac{3}{2} \leq \frac{13}{2} = \delta\left(\frac{11}{2}\right) \quad \text{and} \quad \left| f(3) - \inf_{[2, 6]} f \right| = |4 - 1| = 3 \leq 4 = \delta(3)$$

Therefore, $f\left(\frac{11}{2}\right) =_{\delta} \sup_{[2, 6]} f$ and $f(3) =_{\delta} \inf_{[2, 6]} f$.

7. Comparisons of Classical and Soft Continuity Types

This section compares soft continuity with classical continuity. Moreover, it also compares left / right / upper / lower soft continuity with their classical forms. Throughout this section, let $A \subseteq \mathbb{R}$, $a \in A$, and $f : A \rightarrow \mathbb{R}$ be a function unless stated otherwise.

Theorem 92. If f is classical continuous at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that f is (τ, ε) -soft continuous at a .

Proof. Let f be classical continuous at a . Then,

$$\forall \varepsilon^* > 0, \exists \delta > 0 \ni (x \in (a - \delta, a + \delta) \cap A \Rightarrow |f(x) - f(a)| < \varepsilon^*)$$

Therefore, there exist at least one $\varepsilon^* \geq 0$ and one $\delta > 0$ such that for $\tau(a) = (a - \delta, a) \cup (a, a + \delta)$ and $\varepsilon(a) = \varepsilon^*$, $x \in \tau_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$. Hence, f is (τ, ε) -soft continuous at a . $\square$

Example 27. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 3$, f is classical continuous and $(\tau_\delta, \varepsilon)$ -soft continuous at $a = 2$ such that $\delta(2) = 3$ and $\varepsilon(2) = 4$ because $\lim_{x \rightarrow 2} f(x) = 5 = f(2)$ and $f(2) = 5 \in \text{slim}(f, 2, \tau_\delta, \varepsilon) = [4, 6]$.

The converse of Theorem 92 is not always correct (see Example 28).

Example 28. For the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$, $0 < \delta(2) < 1$, and $\varepsilon(2) \geq 1$, f is $(\tau_\delta, \varepsilon)$ -soft continuous at $a = 2$. Here, $f(2) = 2 \in \text{slim}(f, 2, \tau_\delta, \varepsilon) = [2 - \varepsilon(2), 1 + \varepsilon(2)]$. On the other hand, f is not classical continuous at $a = 2$.

Theorem 93. If f is classical left continuous at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that f is left (τ, ε) -soft continuous at a .

Theorem 94. If f is classical right continuous at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that f is right (τ, ε) -soft continuous at a .

Theorem 95. If f is classical upper semicontinuous at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that f is upper (τ, ε) -soft continuous at a .

Theorem 96. If f is classical lower semicontinuous at a , then there exist functions $\tau : \mathbb{R} \rightarrow P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ such that f is lower (τ, ε) -soft continuous at a .

By setting $\tau(a) = (a - \delta, a)$, $\tau(a) = (a, a + \delta)$, $\tau(a) = (a - \delta, a + \delta)$, and $\tau(a) = (a - \delta, a + \delta)$, respectively, taking $\varepsilon(a) = \varepsilon^*$, and by using $\varepsilon - \delta$ formulations of classical left and right continuity and upper and lower semicontinuity, the proofs can be obtained similar to the proof of Theorem 92.

8. Conclusions

This study establishes a comprehensive framework for the soft limit and soft continuity, central concepts in soft analysis. It redefines the soft limit, upper soft limit, and lower soft limit, together with their soft continuity counterparts, introduces left and right soft limits, soft left and right variances, the soft limit at infinity, and left and right soft continuity, and investigates their fundamental properties. These investigations clarify the relationships among soft limit types and soft variance types, explore boundedness criteria and closed-interval properties for left and right soft limits, and derive key algebraic and order-theoretic rules. Moreover, the paper proves that the soft limit exists if and only if the intersection of left and right soft limits—and similarly of upper and lower soft limits—is nonempty. In the part of soft continuity, it establishes necessary and sufficient conditions by showing that soft continuity is held when both upper and lower soft continuity (and, equivalently, left and right soft continuity) are simultaneously satisfied. In addition, it features characterizations of soft limit types and their soft continuity counterparts and analyzes boundedness conditions and some algebraic rules. Additionally, the paper manifests that the occurrence of classical limit and continuity types entails their soft correspondences.

The study's theoretical contributions provide a robust foundation for extending soft analysis to more complex structures. These results not only substantiate the logical integrity of the soft limit and soft continuity but also pave the way for the development of soft convergence, soft Cauchy sequences, soft limit superior and soft limit inferior, soft summability, soft probability [11–13], soft portfolio control [14], and forecasting by soft sets [15]. Future studies may also focus on investigating these notions in normed and metric spaces, as well as exploring multivariable soft functions and their analytical properties [16]. Overall, this

study lays the groundwork for broadening both the theoretical and practical scope of soft analysis. Some notations in the present paper and their meanings are presented in Table 1.

Table 1. Some notations in the present paper and their meanings.

Notations	Meanings
$\mathbb{N}, \mathbb{R}, \mathbb{R}^-, \mathbb{R}^+, \text{ and } \mathbb{R}^{\geq 0}$	The sets of all nonnegative integers, real numbers, negative real numbers, positive real numbers, and nonnegative real numbers, respectively
$P(U)$	The set of all classical subsets of a set U
$\tau, \lambda, \text{ and } \kappa$	Functions defined from $\mathbb{R}$ to $P(\mathbb{R})$, where $\tau(a)$, $\lambda(a)$, and $\kappa(a)$ are sets of points close to a but not equal to a
$\varepsilon, \alpha, \text{ and } \beta$	Functions defined from $\mathbb{R}$ to $\mathbb{R}^{\geq 0}$
δ	A function defined from $\mathbb{R}$ to $\mathbb{R}^+$
$\Phi(A, B)$	The set of all functions defined from A to B
$\text{Dom}(f)$	The domain set of a function f
$\tau_{D(f)}(a)$	The set $\tau(a) \cap \text{Dom}(f)$
$\tilde{\tau}(a)$	The set $\tau(a) \cup \{a\}$
$\tilde{\tau}_{D(f)}(a)$	The set $\tilde{\tau}(a) \cap \text{Dom}(f)$
$\tau^+(a)$	The set $\tau(a) \cap (a, \infty)$
$\tau^-(a)$	The set $\tau(a) \cap (-\infty, a)$
$\tau_\delta(a)$	The set $[a - \delta(a), a] \cup (a, a + \delta(a)]$
$\tau_\delta^+(a)$	The set $(a, a + \delta(a)]$
$\tau_\delta^-(a)$	The set $[a - \delta(a), a]$
$\tilde{\tau}^+(a)$	The set $\tau^+(a) \cup \{a\}$
$\tilde{\tau}^-(a)$	The set $\tau^-(a) \cup \{a\}$
$\tilde{\tau}_\delta(a)$	The set $[a - \delta(a), a + \delta(a)]$
$\tilde{\tau}_\delta^+(a)$	The set $(a, a + \delta(a)]$
$\tilde{\tau}_\delta^-(a)$	The set $[a - \delta(a), a]$
$\tau_{D(f)}^+(a)$	The set $\tau^+(a) \cap \text{Dom}(f)$
$\tau_{D(f)}^-(a)$	The set $\tau^-(a) \cap \text{Dom}(f)$
$\tilde{\tau}_{D(f)}^+(a)$	The set $\tilde{\tau}^+(a) \cap \text{Dom}(f)$
$\tilde{\tau}_{D(f)}^-(a)$	The set $\tilde{\tau}^-(a) \cap \text{Dom}(f)$
$\text{slim}(f, a, \tau, \varepsilon)$	The set of all (τ, ε) -soft limits of f at a
$\text{slim}^+(f, a, \tau, \varepsilon)$	The set of all upper (τ, ε) -soft limits of f at a
$\text{slim}^-(f, a, \tau, \varepsilon)$	The set of all lower (τ, ε) -soft limits of f at a
$\text{slim}_\tau(f, a, \tau^-, \varepsilon)$	The set of all left (τ, ε) -soft limits of f at a
$\text{slim}_\tau(f, a, \tau^+, \varepsilon)$	The set of all right (τ, ε) -soft limits of f at a
$\Delta(f, a, \tau)$	The soft τ -variance of f at a
$\Delta(f, a, \tau_\delta)$	The soft τ_δ -variance of f at a
$\Delta(f, a, \tau^-)$	The soft τ -left variance of f at a
$\Delta(f, a, \tau_\delta^-)$	The soft τ_δ -left variance of f at a
$\Delta(f, a, \tau^+)$	The soft τ -right variance of f at a
$\Delta(f, a, \tau_\delta^+)$	The soft τ_δ -right variance of f at a
$\leq$	The order relation on intervals, where $[a, b] \leq [c, d]$ iff $a \leq c$ and $b \leq d$
τ_∞	A function defined $\mathbb{R}$ to $P(\mathbb{R})$, where $\tau_\infty(K) \subseteq (K, \infty)$ and $\tau_\infty(K)$ is a set unbounded above
$\tau_{-\infty}$	A function defined $\mathbb{R}$ to $P(\mathbb{R})$, where $\tau_{-\infty}(K) \subseteq (-\infty, K)$ and $\tau_{-\infty}(K)$ is a set unbounded below
$\text{slim}(f, \tau_\infty(K), \varepsilon)$	The set of all $(\tau_\infty(K), \varepsilon)$ -soft limits of f as $x \rightarrow \infty$
$\text{slim}(f, \tau_{-\infty}(K), \varepsilon)$	The set of all $(\tau_{-\infty}(K), \varepsilon)$ -soft limits of f as $x \rightarrow -\infty$
$\Delta(f, \tau_\infty(K))$	The soft $\tau_\infty(K)$ -variance of f
$\Delta(f, \tau_{-\infty}(K))$	The soft $\tau_{-\infty}(K)$ -variance of f
$f(a) =_\varepsilon L$	The inequality $ f(a) - L \leq \varepsilon(a)$ holds

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